TRIGONOMETRY

PART I

INTERMEDIATE TRIGONOMETRY

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PREFACE

This volume, designed as the first part of a complete text-book on Trigonometry, is intended for use in the first-year classes in the Universities and in the more advanced classes in the schools.

In the discussion of the general angle in Chapter I attention is drawn to the proper method of naming the angle. In Chapter II the Circular Functions of the general angle are defined by means of rectangular co-ordinates. For the purposes of Part I only a few very simple formulæ in co-ordinate geometry are needed, and these are given at the beginning of this chapter. In the one or two places later on where, in connection with alternative methods, a further knowledge of the analytical geometry of the straight line is assumed, attention is drawn to the fact; if so desired, these parts may be omitted by the reader.

In Chapter III the relations between the circular functions of angles which differ by a right angle or a multiple of a right angle are established for the general angle. Chapter IV deals with the graphs of the Circular Functions, and the graphical solution of equations.

The proofs of the Addition Theorems in Chapter VI are valid for all angles. These proofs are based on the theory of Orthogonal Projection, of which an account is given in Chapter V. Chapter VII is devoted to transformations of products and sums, summation of series and the solution of equations, and Chapter VIII to the standard linear equation.

Triangle formulæ are discussed in Chapter IX, and the solution of triangles in Chapter X, special attention being paid to methods of arranging the calculations. The last two chapters are concerned with heights and distances, and the properties of quadrilaterals.

Large collections of examples, with answers, are given at the ends of the chapters.

It has been thought better not to print sets of logarithmic and trigonometric tables in the book, as it is usually more convenient for the student, when solving problems, to work with separate sets of tables. The solutions of the examples given in the book are based on five-figure tables; but, if preferred, four-figure tables may be used.

The authors desire to express their cordial thanks to Dr. George Thomson and Mr. Albert Anderson for their valuable assistance in the work of proof-reading, and, more especially, for their help in the arduous task of checking the answers to the examples.

T. M. M.
W. A.

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TRIGONOMETRY

PART I

INTERMEDIATE TRIGONOMETRY

CHAPTER I

ANGLES; MEASUREMENT OF ANGLES; ARCS; SECTORS

§ 1. The Angles made by One Straight Line with Another

LET OA and OB (Fig. 1) be two fixed straight lines drawn from O. A variable straight line OP, with one end at O, which starts from a position along OA, turns about O in the plane AOB and finally takes up a position along OB, is said to describe an angle AOB, or *an angle which* OB *makes with* OA. Such an angle is represented by $A\hat{O}B$ or $\angle AOB$. The revolving line OP is called a *radius vector*. OA is the *initial arm* and OB the *final arm* of $\angle AOB$.

In describing $\angle AOB$ the radius vector may revolve in either sense, or first in one sense and then in the other: all that is essential is that the initial position should be along OA, and the final position along OB. *The order of the letters in the symbol for the angle is of fundamental importance*, the symbol $\angle BOA$ representing an angle described by a radius vector in turning about O from an initial position along OB to a final position along OA, that is, *an angle which* OA *makes with* OB.

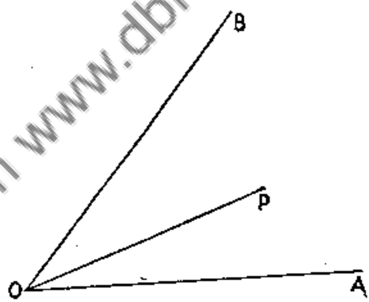

FIG. 1.

Convention as to Sign.—An angle described by a radius vector revolving in the counter-clockwise direction, that is, in the sense opposite to that in which the hands of a clock appear to move, is conventionally reckoned positive.

An angle described in the clockwise sense is reckoned negative.

Coterminal Angles.—Angles which have the same initial arm and the same final arm are called coterminal. Thus the symbol $\angle$AOB may represent any one of an infinite set of coterminal angles, any two of which differ by a number of complete revolutions of the radius vector, that is, by a multiple of four right angles. If α is any selected angle of this set, all the angles of the set are given by the expression $\alpha + (4n$ right angles), where n is zero or any integer, positive or negative.

Coterminal angles have so many properties in common that it is usually unnecessary to specify which particular

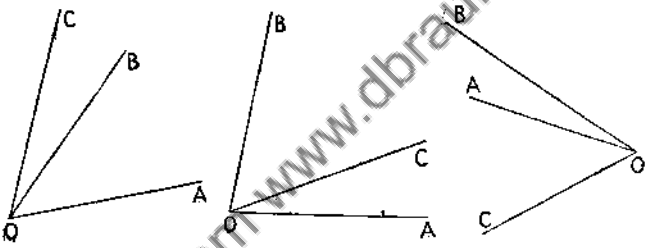

FIG. 2.

angle is represented by the symbol $\angle$AOB. When no ambiguity is likely to arise, the geometrically smallest angle may be taken.

Addition of Angles.—Suppose that OA, OB, OC (Fig. 2) are any three concurrent and coplanar straight lines. A radius vector, in turning about O from the direction OA to the direction OB describes $\angle$AOB; and, in turning, further, from the direction OB to the direction OC, it describes $\angle$BOC. Since its initial and final directions are respectively OA and OC, it describes in all $\angle$AOC. Hence, *for all relative positions of* OA, OB, OC,

$$\angle AOB + \angle BOC = \angle AOC. \quad . \quad . \quad (1)$$

Again, if OD is any line drawn from O in the plane of OA, OB, OC, it follows from (1) that

$$\angle AOB + \angle BOC + \angle COD = \angle AOC + \angle COD = \angle AOD;$$

and the result can be extended to any number of angles.

Equation (1) means that, if to any one of the angles represented by the symbol $\angle AOB$ there is added any one of the angles represented by the symbol $\angle BOC$, the resulting angle is one of those represented by the symbol $\angle AOC$. This holds for angles of any size and of either sign.

Again, (1) may be written in the form

$$\angle BOC = \angle AOC - \angle AOB,$$

which shows that any angle may be expressed as the difference of two angles which have a common initial arm drawn in any arbitrary direction in the plane of the given angle.

§ 2. Measurement of Angles

There are two systems of measurement of angles which are of importance, the sexagesimal and the circular system.

Sexagesimal System.—In the sexagesimal system the geometrical unit, the right angle, is subdivided into 90 equal parts called *degrees*. Thus in one complete revolution a radius vector describes an angle of 360 degrees. The degree is subdivided into 60 equal parts called *minutes*, and the minute into 60 equal parts called *seconds*. The notation for these units is made clear by the example $63° \ 54' \ 21''$, which represents 63 degrees 54 minutes 21 seconds.

This system of measurement is used in most trigonometrical calculations.

Example 1.—Show that $23 \cdot 72° = 23° \ 43' \ 12''$, and that $48° \ 17' \ 29'' = 48 \cdot 2914°$, approximately.

Example 2.—If OX, OP, OQ are in the same plane, and if OP and OQ make with OX angles α and $-\beta$, respectively, show that $\angle QOP = \alpha + \beta$.

$$\angle QOP = \angle XOP - \angle XOQ = \alpha - (-\beta) = \alpha + \beta.$$

Example 3.—OP makes $-57°$ with OX and $153°$ with OQ; OR makes $312°$ with OQ and $-45°$ with OS, all these lines being in the same plane. Find the angle which OS makes with OX.

It is given that $\angle XOP = -57°$, $\angle QOP = 153°$; $\angle QOR = 312°$, $\angle SOR = -45°$.

$$\begin{aligned}\text{Hence } \angle XOS &= \angle XOP + \angle POQ + \angle QOR + \angle ROS \\ &= -57° - 153° + 312° + 45°, \text{ since} \\ &\quad \angle POQ = -\angle QOP, \\ &\quad \angle ROS = -\angle SOR, \\ &= 147°.\end{aligned}$$

Circular System.—Before the unit angle in the circular system of measurement can be defined, it is necessary to consider the definition of the length of an arc of a circle. This definition involves the notion of a limit.

Limit of a Sequence.—If a set of numbers

$$u_1, u_2, u_3, \ldots, u_n, \ldots$$

is arranged to correspond to the set of integers

$$1, 2, 3, \ldots, n, \ldots,$$

it is called an *infinite sequence* or simply a *sequence*, and is denoted by (u_n). The quantities u_1, u_2, u_3, . . . are called the *elements* of the sequence. A sequence (u_n) is said to be convergent if there is a number l such that the difference between u_n and l can be made as small as we please by taking n sufficiently large. It is then said that the sequence *converges to the limit* l, or that u_n *tends to* l *as* n *tends to infinity*. This may be expressed symbolically by writing $u_n \to l$ as $n \to \infty$, the arrow-head denoting "tends to"; or by the equation $\underset{n\to\infty}{L} u_n = l$, which is read "the limit of u_n when n tends to infinity is l."

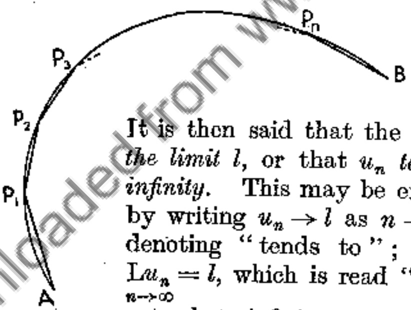

FIG. 3.

Length of an Arc.—Now let AB (Fig. 3) be an arc of a curve, and let P_1, P_2, . . ., P_n be n points taken in order on the arc. *The length of the arc* AB is defined as the limit of the sum of the lengths of the chords AP_1, P_1P_2, P_2P_3, . . ., P_nB as n tends to infinity, the length of each chord tending at the same time to zero.

THEOREM.—*If two arcs of circles subtend equal angles at their centres, they bear the same ratio to their radii.* . (2)

Let AB and ab (Fig. 4) be arcs of two circles whose centres are C and c and whose radii are R and r, respectively, such that $\angle ACB = \angle acb$.

It is required to prove that $\dfrac{arc\ AB}{R} = \dfrac{arc\ ab}{r}$.

Let $P_1, P_2, \ldots, P_n$ be n points taken in order on the arc AB, and let the n points $p_1, p_2, \ldots, p_n$ be so taken on the arc ab that

$$\angle ACP_1 = \angle acp_1, \angle P_1CP_2 = \angle p_1cp_2, \ldots, \angle P_{n-1}CP_n = \angle p_{n-1}cp_n.$$

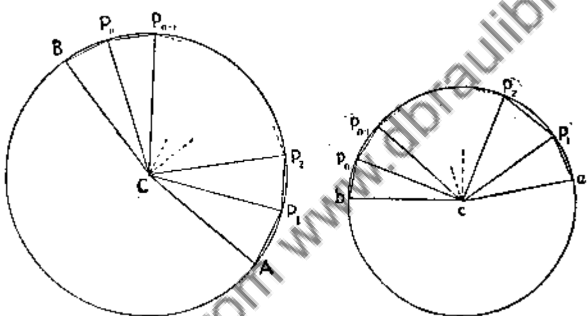

FIG. 4.

It then follows from the equality of $\angle ACB$ and $\angle acb$ that $\angle P_nCB = \angle p_ncb$.

Thus the triangles $ACP_1, P_1CP_2, \ldots, P_nCB$ are respectively similar to the triangles $acp_1, p_1cp_2, \ldots, p_ncb$; so that

$$\frac{AP_1}{R} = \frac{ap_1}{r};\ \frac{P_1P_2}{R} = \frac{p_1p_2}{r};\ \ldots;\ \frac{P_nB}{R} = \frac{p_nb}{r}.$$

Therefore

$$\frac{AP_1 + P_1P_2 + \ldots + P_nB}{R} = \frac{ap_1 + p_1p_2 + \ldots + p_nb}{r}.$$

Now let n tend to infinity, and let each of the angles ACP_1, $P_1CP_2, \ldots, P_nCB$ tend to zero. Then the sum of the chords $AP_1, P_1P_2, \ldots, P_nB$ tends to the arc AB, and

P and Q ; and let the length of the arc PQ, which subtends $\angle$AOB at the centre, be l.

If now R is a point on the circle such that the arc PR is equal to the radius

$$\frac{\angle \text{AOB}}{1 \text{ radian}} = \frac{\angle \text{POQ}}{\angle \text{POR}} = \frac{\text{arc PQ}}{\text{arc PR}},$$

since angles at the centre of a circle are proportional to the arcs on which they stand.

Hence
$$\theta = \frac{l}{r}. \qquad \qquad (3)$$

Thus the circular measure of any angle is equal to the ratio which the length of an arc of a circle subtending that angle at the centre bears to the radius of the circle.

Length of an Arc of a Circle.—Written in the form $l = r\theta$, equation (3) gives the length of an arc of a circle as the product of the radius and the number of radians in the angle which the arc subtends at the centre.

Relation between Circular and Sexagesimal Measure.—Since the ratio of the whole circumference of the circle, subtending four right angles at the centre, to the radius is $2\pi r/r = 2\pi$, it follows that :

$$\begin{aligned} 2\pi \text{ radians} &= 4 \text{ right angles} = 360^\circ, \\ \pi \text{ radians} &= 2 \text{ right angles} = 180^\circ, \\ \tfrac{1}{2}\pi \text{ radians} &= 1 \text{ right angle} \;\; = \;\; 90^\circ, \end{aligned}$$

and that
$$1 \text{ radian} = \frac{180^\circ}{\pi}.$$

Example 4.—Show that 1 radian = 57° 17′ 45″, correct to the nearest second.

Again, if D and θ are respectively the number of degrees and the number of radians in a given angle,

$$\frac{\text{D}}{180} = \frac{\theta}{\pi} \qquad \qquad (4)$$

since each member of the equation is equal to the ratio of the given angle to two right angles. Equation (4) may be used to change from one system of measurement to the other.

When no symbol of the sexagesimal system appears in the statement of the size of an angle, it is understood that the angle is expressed in radians. For example, the angle π, the angle 0·5, mean the angles π radians and 0·5 of a radian, respectively.

Example 5.—Express 37° 27′ in circular measure, and 1·428 radians in sexagesimal measure.

From (4), for an angle of 37° 27′,

$$\frac{\theta}{\pi} = \frac{37 \cdot 45}{180},$$

whence $$\theta \fallingdotseq 0 \cdot 6536.*$$

Also, for an angle of 1·428 radians,

$$\frac{D}{180} = \frac{1 \cdot 428}{\pi},$$

which gives $D \fallingdotseq 81 \cdot 818$. The angle is therefore 81° 49′, approximately.

Example 6.—Calculate the length of an arc of a circle of radius 7 inches which subtends an angle of 37° 27′ at the centre.

The number of radians in 37° 27′ is 0·6536 (*Example* 5).

Hence, by (3), the length of the arc is $7 \times 0 \cdot 6536$, or approximately 4·575 inches.

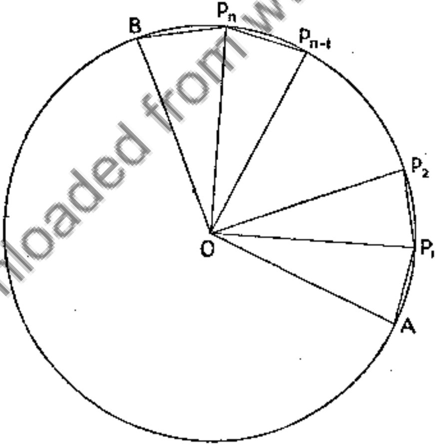

FIG. 6.

§ 3. Area of a Sector of a Circle

Let the angle AOB (Fig. 6) at the centre of a circle of radius r contain θ radians, and let the length of the arc AB be l. Then $l = r\theta$.

* The symbol $\fallingdotseq$ means "is approximately equal to."

Suppose that n points $P_1, P_2, \ldots, P_n$ are taken in order on the arc AB. Then, if S is the area of the sector OAB, the area Σ of the polygon $OAP_1P_2 \ldots P_nBO$ tends to S when n tends to infinity, each of the chords $AP_1, P_1P_2, \ldots, P_nB$ tending at the same time to zero. Now the polygon is the sum of the triangles $OAP_1, OP_1P_2, \ldots, OP_nB$. Hence, if p is the least and P the greatest of the perpendiculars from O to the chords $AP_1, P_1P_2, \ldots, P_nB$,

$$\tfrac{1}{2}p(AP_1 + P_1P_2 + \ldots + P_nB) \leqq \Sigma \leqq \tfrac{1}{2}P(AP_1 + P_1P_2 + \ldots + P_nB).$$

Now let n tend to infinity, each of the chords $AP_1, P_1P_2, \ldots, P_nB$ tending at the same time to zero. Then p and P both tend to r, and the sums within the brackets tend to l. It follows that Σ must tend to $\frac{1}{2}rl$, so that

$$S = \tfrac{1}{2}rl = \tfrac{1}{2}r^2\theta. \qquad (5)$$

The area of a sector of a circle is thus equal to the area of a triangle whose base is equal in length to the arc of the sector, and whose altitude is equal to the radius of the sector.

The complete circle may be considered as a sector whose angle is 2π. Hence by (5) the area of the circle is πr^2, or * $\frac{1}{4}\pi d^2$, where d is the diameter.

Example.—Calculate the area of a sector of a circle of diameter 12 inches whose angle is 97° 19′.

The number of radians in the angle is 1·6985. Therefore, by (5), the area of the sector is

$$\tfrac{1}{2} \times 6^2 \times 1{\cdot}6985 \text{ or } 30{\cdot}57 \text{ square inches.}$$

§ 4. Points of the Compass

On a circle on the card of the mariner's compass 32 equidistant points are marked, as shown in Fig. 7. The *cardinal points* are North, South, East, West, denoted by N., S., E., W., respectively. The points halfway between these are North-East (N.E.), North-West (N.W.), South-East (S.E.) and South-West (S.W.). Midway between these 8 points

* Since $\frac{1}{4}\pi \doteqdot 0{\cdot}7854$, this gives the blacksmith's rough working rule; "take four-fifths of the square on the diameter."

are the points North-North-East (N.N.E.), North-North-West (N.N.W.), etc., and midway between the 16 points are other 16, North by East (N. by E.), North-East by North (N.E. by N.), etc. The angle between two adjacent points is sometimes spoken of as a point; its value is $11\frac{1}{4}°$.

Bearings.—If the card is laid horizontally so that the line SN runs from South to North, the radius of the circle which is in the direction towards any point gives the bearing

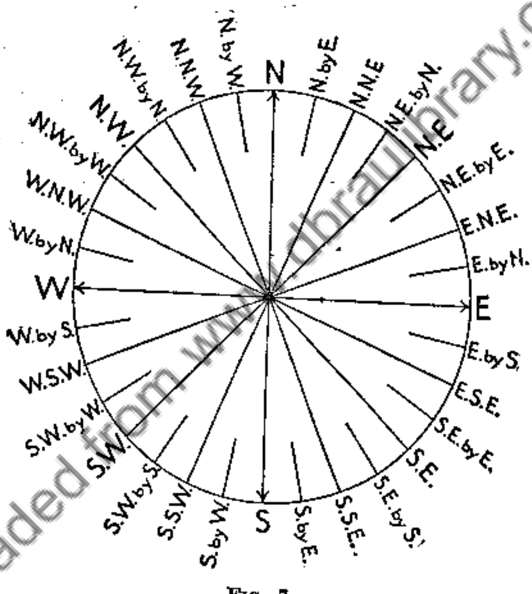

FIG. 7.

of that point from the centre of the circle. The bearing is usually given in terms of the angle between this radius and the radius to one of the cardinal points. For example, if O (Fig. 8) is the centre of the circle, a direction OP between OE and ON which makes an angle 37° 14′ with OE is said to be 37° 14′ north of east or 52° 46′ east of north. These bearings are sometimes denoted by E. 37° 14′ N. and N. 52° 46′ E., the angular measurements being made from the direction which comes first towards that which comes last.

The bearing is also frequently measured from the north in the clockwise direction. For example, in this system the bearing W. 17° N. becomes 287°.

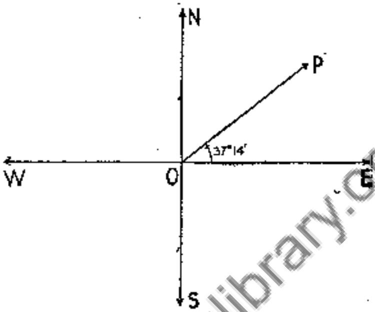

FIG. 8.

Longitude and Latitude.—The earth is approximately spherical in shape. Let it be regarded as a perfect sphere. Let O, N, S (Fig. 9) be the centre of the earth, the North Pole and the South Pole respectively; then SN is the earth's axis. Any semi-circle with O as centre and N and S as its extremities is called a *meridian*; for example, NPMS is the meridian of the point P on the surface. The circular section of the surface by the plane through O at right angles to the axis is called the *Equator*.

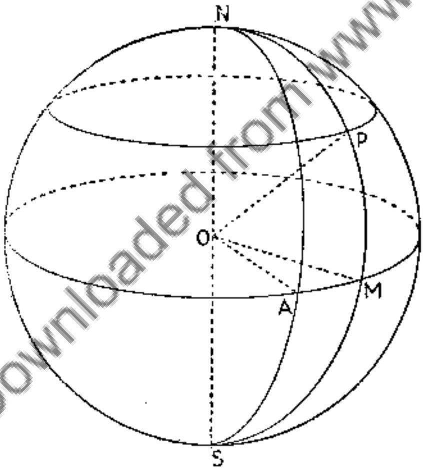

FIG. 9.

If the meridian which passes through Greenwich cuts the equator in A, the *longitude* of P is the angle AOM subtended at the centre by the arc AM of the equator intercepted between the meridian of Greenwich and the meridian of P, the direction, east or west, from A to M being specified.

The *latitude* of P is the angle MOP subtended at the centre by the arc MP of the meridian of P intercepted between the equator and P, the direction, north or south, from M to P being specified.

The latitude and the longitude of a point together define the position of the point.

Circles on the surface parallel to the equator are called parallels of latitude.

EXAMPLES I

In *Examples* 1-4 it is assumed that all the lines are in the same plane.

1. If $\angle AOB = 65°$, $\angle BOC = -23°$, $\angle COD = 142°$ and $\angle DOE = -155°$, find (i) the angle which OE makes with OA, and (ii) the angle which OA makes with OE. Apply an approximate check to your results by drawing an accurate diagram, setting off the angles by means of a protractor.

Ans. (i) 29°; (ii) − 29°.

2. If OB makes − 168° with OA, OA makes 75° with OC, and OD makes 107° with OB, find (i) the least positive angle which OC makes with OD, and (ii) the numerically least negative angle which OA makes with OD.

Ans. (i) 346°; (ii) − 299°.

3. $\angle XOA = 30°$, $\angle OAB = -105°$, $\angle ABC = -34°$ and $\angle BCD = -136°$. Find the angle which (i) CD, (ii) DC makes with OX.

Ans. (i) − 65°; (ii) 115°.

4. $\angle XOA = -142°$, $\angle OAB = 27°$, $\angle ABC = 58°$ and $\angle BCD = -16°$. Find the angles which AB, BC and CD make with OX.

Ans. 65°; − 57°; 107°.

5. Express the following angles in circular measure:
(i) 5° 19′; (ii) 34° 47′; (iii) 57° 16′; (iv) 116° 32′; (v) 172°; (vi) 325° 57′; (vii) 1′; (viii) 1″.

Ans. (i) 0·09279; (ii) 0·60708; (iii) 0·99949; (iv) 2·03389; (v) 3·00197; (vi) 5·68890; (vii) 0·00029; (viii) 0·000005.

6. The following angles are given in circular measure. Express them in sexagesimal measure, correct to the nearest tenth of a minute:

(i) 0·5; (ii) 0·63714; (iii) 1·49891; (iv) 3·12617; (v) $\frac{3}{7}\pi$; (vi) 2·3268π.

Ans. (i) 28° 38·9′; (ii) 36° 30·3′; (iii) 85° 52·9′; (iv) 179° 7·0′; (v) 77° 8·6′; (vi) 418° 49·4′.

7. Find, to the nearest second, the time taken by (i) the minute hand, (ii) the hour hand of a clock to turn through 1 radian.

Ans. (i) 9 mins. 33 secs.; (ii) 1 hr. 54 mins. 35 secs.

8. Calculate the circular measure of the angle between the hands of a clock when the time is (i) 10 h. 13 m.; (ii) 10 h. 23 m.

Ans. (i) 2·2951; (ii) 3·0281.

9. The acute angles of a right-angled triangle differ by 0·5 of a radian. Find these angles, correct to the nearest half minute.

Ans. $59° 19\frac{1}{2}'$: $30° 40\frac{1}{2}'$.

10. Two of the angles of a triangle exceed the third by 0·4 of a radian and 1 radian, respectively. Find the angles of the triangle, correct to the nearest half-minute.

Ans. $33° 15\frac{1}{2}'$; $56° 11'$; $90° 33\frac{1}{2}'$.

11. Find the angle subtended at the centre of a circle of radius 5 feet by an arc 9 inches long.

Ans. $8° 35·7'$.

12. Considering the Equator as a circle of diameter 7926·7 miles, calculate its circumference.

Ans. 24902 miles.

13. Calculate the angle subtended at the centre of the earth by a 10-mile arc of the Equator.

Ans. $8' 40''$.

14. Calculate the length of an arc of the Equator which subtends an angle of 10° at the centre of the earth.

Ans. 691·7 miles.

15. Considering the earth as a sphere of diameter 7913·3 miles, calculate the distance along a meridian from the North Pole to a point in latitude (i) 55° 49′ N.; (ii) 55° 49′ S.

Ans. (i) 2360·6 miles; (ii) 10069·6 miles.

16. Calculate the distance along a meridian from a point in latitude 28° 11′ N. to (i) the Equator; (ii) the North Pole; (iii) the South Pole.

Ans. (i) 1946 miles; (ii) 4269 miles; (iii) 8161 miles.

17. London is 343 miles from Glasgow. Find to the nearest minute the angle subtended at the centre of the earth by the line joining these towns.

Ans. 4° 58′.

18. Calculate the distance on the earth's surface between two places on the same meridian whose latitudes are 43° 57′ N. and 25° 16′ S.

Ans. 4779·9 miles.

19. Calculate the difference in latitude of two places, one of which is 350 miles south of the other.

Ans. 5° 4′.

20. Assuming that the earth's orbit is a circle of radius 95×10^6 miles, show that its speed is about 19 miles per second.

21. How many revolutions per minute does a bicycle wheel of diameter (i) 26 inches; (ii) 28 inches make when the cyclist's speed is 10 miles per hour?

Ans. (i) 129·3; (ii) 120·0.

22. At what speed is a cyclist travelling when the 26-inch diameter wheels of his machine make 150 revolutions per minute?

Ans. 11·60 miles per hour.

23. The pedal and the hub sprocket wheels of a fixed wheel bicycle have 48 and 18 teeth, respectively; and the diameter of the rear wheel is 26 inches. Calculate (i) the distance travelled by the bicycle for one complete turn of the pedals; and (ii) the speed of the bicycle when the pedals are revolving at the rate of 75 revolutions per minute.

Ans. (i) 18 feet 2 inches; (ii) 15·47 miles per hour.

24. A train travels at 50 miles per hour on an arc of a circle of radius 4000 yards. Show that it changes direction at the rate of 21′ per second.

25. Calculate the area of a sector of a circle in the following cases, the first value in each case being the radius of the circle, the second the angle of the sector:

(i) 10 inches; 47° 23′; (ii) 7 feet; 104° 32′;

(iii) 25 cms.; $\frac{2\pi}{3}$; (iv) 17·43 inches; 22° 4′;

(v) 36·53 inches; 61° 55′; (vi) 9·79 inches; $\frac{3\pi}{7}$.

Ans. (i) 41·35 square inches; (ii) 44·70 square feet; (iii) 654·5 sq. cms.; (iv) 58·50 square inches; (v) 721·0 square inches; (vi) 64·52 square inches.

26. A sector of a circle whose diameter is 13 inches is equal in area to a sector of a circle whose diameter is 17 inches. The

angle of the first sector is 125° 17′. Calculate the angle of the second sector, correct to the nearest minute.

Ans. 73° 16′.

27. Each of three equal circles of radius r touches the other two. Show that the area enclosed between them is approximately $0{\cdot}1613\,r^2$.

28. Calculate the area of a segment of a circle of radius 10 inches which is cut off by a chord of length 10 inches.

Ans. 9·06 square inches.

29. Two sectors of circles are equal in area. Their angles are 97° 44′ and 82° 51′. If the radius of the first is 5 feet, calculate the radius of the second.

Ans. 5·431 feet.

30. Find, to the nearest half-minute, the angle of a sector of a circle if the perimeter of the sector is equal to (i) half the circumference, (ii) the circumference of the circle.

Ans. (i) 65° $24\frac{1}{2}$′; (ii) 245° $24\frac{1}{2}$′.

31. The perimeter of a sector of a circle of radius 9 inches is 30 inches. Find the area of the sector, and, to the nearest minute, the angle of the sector.

Ans. 54 square inches; 76° 24′.

32. The perimeter of a sector of a circle of variable radius r has the constant value p. Show that the area of the sector is $r(\frac{1}{2}p - r)$, and that the sector has maximum area when its angle is 2 radians, or, approximately, 114° $35\frac{1}{2}$′.

CHAPTER II

THE CIRCULAR FUNCTIONS

§ 1. Rectangular Co-ordinates

Representation of Points on a Directed Line.—Let X′OX (Fig. 1) be a line extending indefinitely in both directions. One direction X′OX is chosen as the positive direction and is marked by an arrow-head. A point O on the line is selected as the origin, and a point A_1, on the positive side of O, as the unit point. If distances A_1A_2, A_2A_3, . . ., each equal to OA_1, are marked off on the positive side of O, and distances OA_{-1}, $A_{-1}A_{-2}$, . . ., each equal to A_1O, on the negative side of O, the complete system of integers

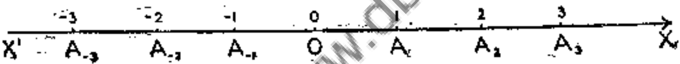

FIG. 1.

. . . -2, -1, 0, 1, 2, 3, . . . is represented by the corresponding points . . . A_{-2}, A_{-1}, O, A_1, A_2, A_3, . . ., respectively. Any other real number p is represented by that point P on the line which is such that $OP/OA_1 = p$. Thus to each point on the line there corresponds one and only one real number; and, conversely, to each real number there corresponds one, and only one, point on the line. We take OA_1 to be the unit of length, and then OP is of length p.

Co-ordinate Axes.—Now take two directed lines X′OX, Y′OY (Fig. 2), intersecting at a common origin O, with assigned unit points A_1 and B_1. Through any point P in the plane of these lines let straight lines PM and PN be drawn parallel to Y′OY and X′OX to meet X′OX and Y′OY in M and N, respectively. Then if $OM = x$, $ON = y$, x and y are called the *co-ordinates* of P, x being the *abscissa* and y the *ordinate*, and P is the point (x, y). X′OX is called the x-axis, or the axis of abscissæ; Y′OY the

y-axis, or the axis of ordinates. To any point P in the plane there corresponds one and only one pair of co-ordinates; and, conversely, to any pair of co-ordinates there corresponds one and only one point in the plane. It should be noted that $NP = x$, $MP = y$, the lengths and directions of these lines being the same as those of OM and ON, respectively.

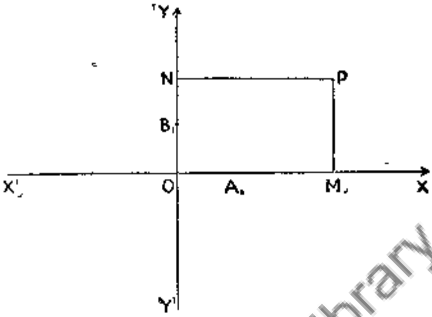

Fig. 2.

In what follows it will be assumed that the angle XOY is a positive right angle: the system of co-ordinates is then said to be *rectangular*. It will be assumed in this chapter that the unit lengths OA_1 and OB_1 are equal.

The part of the co-ordinate plane above OX and to the right of OY is called the first quadrant, the part above OX′ and to the left of OY is the second quadrant, the part below OX′ and to the left of OY′ the third quadrant, and the part below OX and to the right of OY′ the fourth quadrant. At a point in the first quadrant x and y are both positive, in the second quadrant x is negative, y positive, in the third quadrant both are negative, and in the fourth x is positive and y negative.

Lines Parallel to the Axes.—For all points (x, y) on a straight line parallel to the y-axis, x has a constant value, a say. The equation $x = a$, which holds for all points on the line, and for no other points, is called *the equation of the line.* The line lies to the right or left of the y-axis according as a is positive or negative, while, if a is zero, the line is the y-axis.

Similarly the equation of a line parallel to the x-axis is of the form $y = b$, where b is a constant.

Lines through the Origin.—For all points (x, y) on a line through the origin, it follows from the properties of similar triangles that the ratio of y to x has a constant value, k say. If the line is in the first and third quadrants, k is positive; while, if it is in the second and fourth quadrants, k is negative. The equation $y = kx$, which is satisfied by the co-ordinates of all points on the line (including those of the origin), and by those of no other points, is the equation of the line.

Circles with the Origin as Centre.—For all points (x, y) on a circle with the origin as centre and radius r it follows from the theorem of Pythagoras that $x^2 + y^2 = r^2$. This equation, which is not satisfied by the co-ordinates of any other points, is the equation of the circle.

§ 2. Definitions

Let X′OX and Y′OY (Fig. 3) be a pair of rectangular co-ordinate axes, the unit of length being the same on each, and let a radius vector of length r start from a position along the positive part of the x-axis and turn about O through any angle θ to the position OP, where P is the point (x, y). The length r of the radius vector is always positive.

The ratios shown in the following table are called the *circular functions* of θ, or the trigonometrical ratios of θ:

Ratio.	*Name.*	*Notation.*
$\frac{x}{r}$	The cosine of θ	$\cos\theta$
$\frac{y}{r}$	The sine of θ	$\sin\theta$
$\frac{y}{x}$	The tangent of θ	$\tan\theta$
$\frac{r}{x}$	The secant of θ	$\sec\theta$
$\frac{r}{y}$	The cosecant of θ	$\operatorname{cosec}\theta$
$\frac{x}{y}$	The cotangent of θ	$\cot\theta$

Note 1.—These definitions include as particular cases the usual right-angled triangle definitions of the trigonometric ratios of a positive acute angle. If $0 < \theta < 90°$, x and y are positive; and, if M is the foot of the perpendicular from P to OX, $\angle MOP = \angle XOP = \theta$. Hence, from the above definitions,

$$\cos M\hat{O}P = \frac{x}{r} = \frac{\text{adjacent side}}{\text{hypotenuse}},$$

$$\sin M\hat{O}P = \frac{y}{r} = \frac{\text{opposite side}}{\text{hypotenuse}}, \text{ etc.}$$

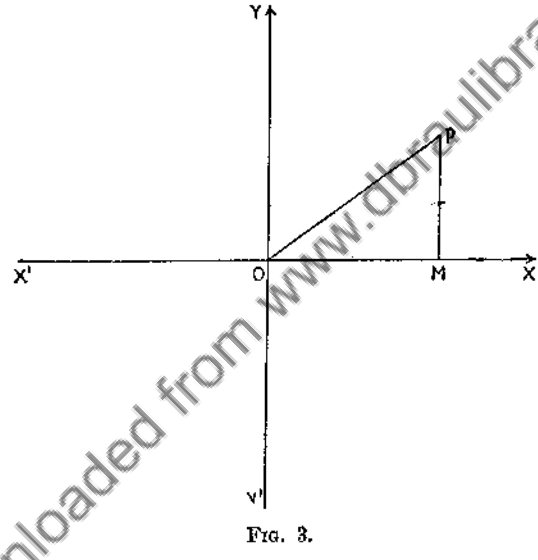

FIG. 3.

Note 2.—The circular functions are independent of the length of the radius vector, and depend only on the value of θ, that is, on the final direction of the radius vector. To show this, let $\angle XOP' = \theta = \angle XOP$, P′ being the point (x', y') and OP′ having length r'. Then P′ must lie on OP or on OP produced. Hence, if M′ is the foot of the perpendicular from P′ to the x-axis, the triangles OMP, OM′P′ are similar, OM and OM′ have the same sign, and MP and M′P′ have the same sign.

Therefore $\frac{x}{x'} = \frac{y}{y'} = \frac{r}{r'}$, so that $\frac{x}{r} = \frac{x'}{r'}$, $\frac{y}{x} = \frac{y'}{x'}$, etc.

Note 3.—Certain simple relations between circular functions of the same angle follow directly from the definitions. For example

$$\sec\theta = \frac{1}{\cos\theta}, \quad \operatorname{cosec}\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{1}{\tan\theta},$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

The cosine, the sine and the tangent, may be called the *principal* circular functions, their reciprocals, the secant, the cosecant and the cotangent being called *secondary*.

Note 4.—Since * $|x| \leqq r$ and $|y| \leqq r$ for all values of θ,

$$|\cos\theta| \leqq 1, \ |\sin\theta| \leqq 1; \ |\sec\theta| \geqq 1, \ |\operatorname{cosec}\theta| \geqq 1.$$

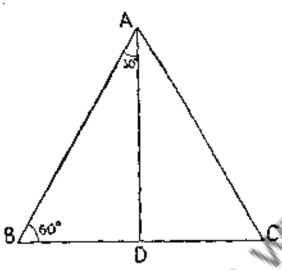

FIG. 4.

There are no restrictions on the values of $\tan\theta$ and $\cot\theta$: by assigning an appropriate value to θ, either may be made to take any real value whatever.

Note 5.—The values of the circular functions of 30° and 60° may be found by drawing the bisector AD (Fig. 4) of the angle A of an equilateral triangle ABC, to meet BC in D. Then $\angle$ADB is right, BD = DC = $\frac{1}{2}$AB, and therefore

$$AD^2 = AB^2 - BD^2 = \tfrac{3}{4}AB^2,$$

so that
$$AD = \frac{\sqrt{3}}{2}AB.$$

Hence, from the triangle ADB,

$$\cos 30^\circ = \frac{\sqrt{3}}{2}; \quad \sin 30^\circ = \frac{1}{2}; \quad \tan 30^\circ = \frac{1}{\sqrt{3}};$$
$$\cos 60^\circ = \frac{1}{2}; \quad \sin 60^\circ = \frac{\sqrt{3}}{2}; \quad \tan 60^\circ = \sqrt{3}.$$

Similarly, from an isosceles right-angled triangle the results

$$\cos 45^\circ = \frac{1}{\sqrt{2}}; \quad \sin 45^\circ = \frac{1}{\sqrt{2}}; \quad \tan 45^\circ = 1$$

are obtained.

Note 6.—The angle XOP is said to be an angle in the first, second, third, or fourth quadrant, according as OP lies in the first, second, third or fourth quadrant. From consideration of

* The symbol $|a|$ denotes the numerical value of a.

the signs of x and y it follows that (i) when θ is in the first quadrant, all the circular functions of θ are positive; (ii) when θ is in the second quadrant, $\sin\theta$ and $\operatorname{cosec}\theta$ are positive, the others negative; (iii) when θ is in the third quadrant, $\tan\theta$ and $\cot\theta$ are positive, the others negative; (iv) when θ is in the fourth quadrant, $\cos\theta$ and $\sec\theta$ are positive, the others negative.

Note 7.—The way in which the circular functions vary as θ varies may conveniently be determined from Fig. 5, in which $\angle XOP = \theta$, P is on the circle with centre the origin and radius unity, MP is the ordinate of P, and Q is the point in which the line OP meets the tangent to the circle at A (1, 0). From the definitions, $\cos\theta = OM$, $\sin\theta = MP$ and $\tan\theta = AQ$; and it is easy to trace the variation of these segments as θ varies.

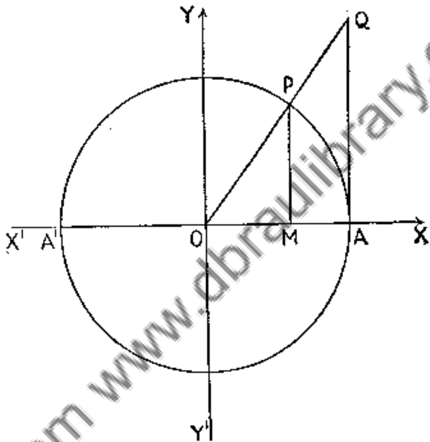

FIG. 5.

The following table shows how the principal circular functions vary as θ increases from 0 to 360°. The arrow sloping upward means "increasing", sloping downward "decreasing". The student should construct a similar table for the secondary circular functions.

θ.	0.	↗	90°.	↗	180°.	↗	270°.	↗	360°.
$\cos\theta$ . .	1	↘	0	↘	-1	↗	0	↗	1
$\sin\theta$ . .	0	↗	1	↘	0	↘	-1	↗	0
$\tan\theta$ . .	0	↗	$\pm\infty$	↗	0	↗	$\pm\infty$	↗	0

It will be noticed that $\tan\theta$ has discontinuities when $\theta = 90°$, 270°, or any angle coterminal with either of these, its value * changing abruptly from $+\infty$ to $-\infty$; for all other values of θ, $\tan\theta$ increases as θ increases.

Similarly, $\cot\theta$ decreases as θ increases, for all values of θ except 0, 180° and coterminal angles. At these values $\cot\theta$ is discontinuous: its value changes abruptly from $-\infty$ to $+\infty$.

Periodicity of the Circular Functions.—A function $f(x)$ is said to be *periodic* if, for all values of x, $f(x+a)=f(x)$, where a is a constant known as a *period* of the function.

Each of the circular functions has the same value for all the angles of a coterminal set, since the position of P, and therefore the co-ordinates of P are the same for all. Hence, for all values of θ, the circular functions of the angles $\theta \pm 360°$, $\theta \pm 2 \times 360°$, $\theta \pm 3 \times 360°$, . . . have the same values as the corresponding functions of θ. The circular functions are therefore periodic, having period 360° or 2π radians. As θ increases from 0 to 360° any specified circular function of θ goes through a set of values in a certain order; as θ increases from 360° to 720°, or from 720° to 1080°, etc., or from $-360°$ to 0, or from $-720°$ to $-360°$, etc., the function goes through the same set of values, in the same order.

It will be seen later (Ch. III, § 1) that $\tan\theta$ and $\cot\theta$ have the smaller period 180° or π radians.

§ 3. The Fundamental Identity

Since the distance of P (Fig. 3) from O is r, and the triangle OMP is right-angled, then, for all values of θ, $x^2+y^2=r^2$. This relation may be expressed in three forms each involving only ratios of x, y and r to one another; namely

$$\left(\frac{x}{r}\right)^2+\left(\frac{y}{r}\right)^2=1, \quad \left(\frac{r}{x}\right)^2=1+\left(\frac{y}{x}\right)^2,$$

and
$$\left(\frac{r}{y}\right)^2=1+\left(\frac{x}{y}\right)^2.$$

* This is a convenient, if somewhat loose way of expressing the fact that as θ, *increasing*, tends to any of the values $90° + k \,.\, 180°$ ($k = 0, \pm 1, \pm 2, \ldots$), $\tan\theta$ increases beyond bound, but as θ, *decreasing*, tends to any of these values, $\tan\theta$ decreases beyond bound. When θ has any of these values, $\tan\theta$ has no value.

These give the three forms of the fundamental identity of Trigonometry: for all values of θ,

$$\cos^2\theta + \sin^2\theta = 1, \quad . \quad . \quad . \quad (1)$$

$$\sec^2\theta = 1 + \tan^2\theta, \quad . \quad . \quad (2)$$

$$\operatorname{cosec}^2\theta = 1 + \cot^2\theta. \quad . \quad . \quad (3)$$

The symbols $\cos^2\theta$, $\sin^2\theta$, etc., are used to denote the squares of $\cos\theta$, $\sin\theta$, etc., in order to avoid the cumbersome notation $(\cos\theta)^2$,* $(\sin\theta)^2$, etc. A similar notation will be used for higher powers.

Identities.—A trigonometric equation is an identity if it is true for all values of the angle or angles involved. A given identity may be established by reducing either side to the other, by reducing each side to the same expression, or by any convenient modification of these methods.

Example 1.—Establish the identity

$$\tan\theta + \cot\theta = \sec\theta\operatorname{cosec}\theta.$$

$$\begin{aligned}\tan\theta + \cot\theta &= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\\ &= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}\\ &= \frac{1}{\cos\theta\sin\theta}, \text{ by (1)}\\ &= \sec\theta\operatorname{cosec}\theta.\end{aligned}$$

Example 2.—Establish the identity

$$(\sec A - \cos A)^2 + (\operatorname{cosec} A - \sin A)^2 - (\cot A - \tan A)^2 = 1.$$

$$\begin{aligned}\text{The left side} &= \sec^2 A - 2 + \cos^2 A + \operatorname{cosec}^2 A - 2 + \sin^2 A\\ &\qquad - \cot^2 A + 2 - \tan^2 A\\ &= (\cos^2 A + \sin^2 A) + (\sec^2 A - \tan^2 A)\\ &\qquad + (\operatorname{cosec}^2 A - \cot^2 A) - 2\\ &= 1,\end{aligned}$$

since, by the fundamental identity, each of the expressions in brackets has the value 1.

Example 3.—Establish the identity

$$\frac{2\sin\alpha}{1 + \cos\alpha + \sin\alpha} = \frac{1 - \cos\alpha + \sin\alpha}{1 + \sin\alpha}.$$

The identity will be true if

* The symbol $\cos\theta^2$ means the cosine of the angle whose measure is θ^2.

$$2\sin\alpha\,(1+\sin\alpha) = (1+\sin\alpha)^2 - \cos^2\alpha. \qquad (a)$$

The right side of $(a) = (1+\sin\alpha)^2 - (1-\sin^2\alpha)$
$= (1+\sin\alpha)(1+\sin\alpha - 1 + \sin\alpha)$
$= 2\sin\alpha\,(1+\sin\alpha)$.

Hence (a) is true, and therefore the given identity is true.

§ 4. Elimination

Suppose that two equations are given, each involving an angle, α say, and other variables. The process of forming from these equations a third equation, true if they are true, and independent of α, is described as eliminating α between the equations. The new equation is called the *eliminant*.

If from the given equations it is possible to derive a pair of equations of the form $\cos\alpha = p$, $\sin\alpha = q$, where p and q are independent of α, the eliminant $p^2 + q^2 = 1$ is given at once by (1). Similarly the second and third forms of the fundamental identity give the eliminant when the given equations reduce to the forms $\sec\alpha = h$, $\tan\alpha = k$, or to $\operatorname{cosec}\alpha = m$, $\cot\alpha = n$.

Again, if the given equations can be combined so as to give the value of one of the circular functions of α, it may be possible to express one of the given equations, or a derived equation, entirely in terms of that circular function. The eliminant is then obtained by substitution.

Example 1.—Eliminate α between the equations

$$x = 2\cos\alpha - \sin\alpha, \quad y = \cos\alpha - 3\sin\alpha.$$

These equations, when solved for $\cos\alpha$ and $\sin\alpha$, give

$$\cos\alpha = \tfrac{1}{5}(3x - y), \qquad \sin\alpha = \tfrac{1}{5}(x - 2y).$$

Therefore, by (1),

$$\tfrac{1}{25}(3x - y)^2 + \tfrac{1}{25}(x - 2y)^2 = 1,$$

or

$$2x^2 - 2xy + y^2 = 5,$$

which is the required eliminant.

Example 2.—Eliminate θ between the equations

$$x\cos\theta + y\sin\theta = a, \qquad x\sin\theta - y\cos\theta = b.$$

Squaring both sides of each equation, and adding corresponding sides of the new equations gives

$$(x^2 + y^2)(\cos^2\theta + \sin^2\theta) = a^2 + b^2,$$

or

$$x^2 + y^2 = a^2 + b^2.$$

Example 3.—Eliminate θ between the equations

$$\text{(i)}\quad x \sec \theta = 1 - y \tan \theta,$$
$$\text{(ii)}\quad x^2 \sec^2 \theta = 5 + y^2 \tan^2 \theta.$$

From (i), $x^2 \sec^2 \theta = 1 - 2y \tan \theta + y^2 \tan^2 \theta$.
Hence, from (ii), $1 - 2y \tan \theta = 5$, or $\tan \theta = -2/y$.
In (ii) put $1 + \tan^2 \theta$ for $\sec^2 \theta$; then substitute $-2/y$ for $\tan \theta$. This gives

$$x^2(1 + 4/y^2) = 5 + 4,$$

or

$$x^2y^2 + 4x^2 - 9y^2 = 0.$$

§ 5. Expression of any Circular Function in terms of any other

Any circular function can be expressed algebraically in terms of any specified circular function of the same angle.

Example 1.—Express the other circular functions of θ in terms of cosec θ.

Let $c = \operatorname{cosec} \theta$. Then $\sin \theta = \dfrac{1}{c}$.

From (3), $\cot^2 \theta = c^2 - 1$; therefore $\cot \theta = \pm \sqrt{(c^2 - 1)}$.

From (1), $\cos^2 \theta = 1 - \dfrac{1}{c^2}$; therefore $\cos \theta = \pm \dfrac{\sqrt{(c^2 - 1)}}{c}$.

Hence $$\tan \theta = \frac{1}{\cot \theta} = \pm \frac{1}{\sqrt{(c^2 - 1)}},$$

and $$\sec \theta = \frac{1}{\cos \theta} = \pm \frac{c}{\sqrt{(c^2 - 1)}}.$$

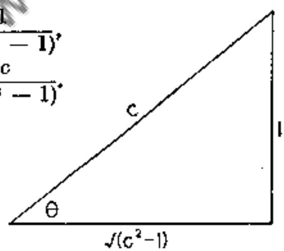

FIG. 6.

The ambiguity in sign in all these cases except the reciprocal arises from the fact that the assigning of a definite value c to cosec θ does not fix definitely the quadrant in which θ lies. If c is positive, θ may be in the first or in the second quadrant: in the former case the plus sign applies, in the latter the minus. If c is negative, θ may be in the third or in the fourth quadrant: in the former case the plus sign must be taken, in the latter the minus.

In the above example the results might have been obtained quickly by means of the following convenient working rule, which can easily be modified to suit other cases.

Sketch roughly a right-angled triangle (Fig. 6) and label one of the angles θ. Label the hypotenuse c and the side opposite to θ unity, so that cosec $\theta = c$; then the remaining side must be $\sqrt{(c^2 - 1)}$. The values of the other circular functions of θ can then be read off at once from the diagram, *the $\pm$ sign being taken with* $\sqrt{(c^2 - 1)}$.

Approximate Construction of Angles when one Circular Function is given.—The approximate construction, using squared paper and a straight edge, of the angles for which a given circular function has a specified value, forms a useful exercise on the definitions of the circular functions.

Example 2.—Using ruler and squared paper, construct the angles whose secants have the value $-97/72$.

Let $\angle XOP$ be one of the required angles, OP having length r and P being the point (x, y).

Then $$\sec X\widehat{O}P = \frac{r}{x} = -\frac{97}{72}.$$

Now r may have any positive value; for convenience take $r = 97$, whence $x = -72$. The ordinate of P is then found from the equation $x^2 + y^2 = r^2$, which gives

$$y^2 = (r + x)(r - x) = 25 \times 169, \text{ or } y = \pm 65.$$

There are thus two possible positions of P, namely

$$P_1(-72, 65) \text{ and } P_2(-72, -65).$$

If a suitable unit is chosen and the points P_1, P_2 plotted and joined to O, then $\angle XOP_1$ and $\angle XOP_2$ are the required angles.

§ 6. On the Use of Tables of Circular Functions

It will be assumed that the student has at his disposal tables of logarithmic and trigonometric functions, and that he is familiar with the use of the former. The examples in this book are based on five-figure tables in which the circular functions and their logarithms are tabulated for angles at 6′ intervals, with mean differences at intervals of 1′; but, if preferred, four-figure tables may be employed, with, of course, a corresponding decrease in the accuracy of the results. The tables of mean differences for the circular functions and their logarithms are to be used in the same manner as in tables of logarithms, apart from the following exceptions, which should be carefully noted. Since, for angles between 0 and 90°, the cosine, the cosecant and the cotangent, and their logarithms, decrease as the angle increases, in using the tables of these functions *mean* *differences corresponding to an increase in the angle should be subtracted.*

The following examples illustrate the methods of reading from the tables:

Example 1.—From the tables find sin 29° 44′ and cot 57° 22′.

From the table of sines, sin 29° 42′ = 0·49546, and the corresponding mean difference for 2′ is 0·00051. Therefore

$$\sin 29° 44' = 0{\cdot}49597.$$

From the table of cotangents, cot 57° 18′ = 0·64199, and the corresponding mean difference for 4′ is 0·00164. Therefore

$$\cot 57° 22' = 0{\cdot}64035.$$

Example 2.—Find from the tables, for the range from 0 to 90°, (i) the angle whose cosine is 0·32153; and (ii) the angle whose secant is 1·34722.

(i) The given value of the cosine = 0·32153.
From the cosine table, cos 71° 18′ = 0·32061.
Difference = 0·00092.
The nearest mean difference = 0·00083,

which corresponds to 3′.

Hence, to the nearest minute, 0·32153 = cos 71° 15′.

(ii) The given value of the secant = 1·34722.
From the secant table, sec 42° 0′ = 1·34563.
Difference = 0·00159.
The nearest mean difference = 0·00145,

which corresponds to 4′.

Hence, to the nearest minute, 1·34722 = sec 42° 4′.

Example 3.—Find from the tables (i) the value of log cosec 67° 27′; and (ii) the angle between 0 and 90° the logarithm of whose tangent is 0·51957.

(i) From the table of logarithms of cosecants,
log cosec 67° 24′ = 0·03470,
and the corresponding mean difference for 3′ = 0·00016.
Therefore log cosec 67° 27′ = 0·03454.

(ii) If θ represents the required angle, log tan θ = 0·51957.
From the table of logarithms of tangents,
log tan 73° 6′ = 0·51738.
Difference = 0·00219.
The nearest mean difference = 0·00232,

which corresponds to 5′.

Hence, to the nearest minute, $\theta = 73° 11'$.

After some practice in reading angles from the tables correct to the nearest minute, the student should practise reading angles correct to the nearest half-minute, taking half of the mean difference for one minute as the mean difference for a half-minute. For instance, in *Example* 2 (ii), above, 0·00159 exceeds the nearest tabulated difference (0·00145) by 0·00014. Hence, as the difference corresponding to $\frac{1}{2}'$ is 0·00018, the required angle is, to the nearest half-minute, $42° 4\frac{1}{2}'$.

It will be noted that in some of the trigonometric tables there are certain ranges of the angle for which, owing to the rapid increase of the mean differences, these are not tabulated. In using these parts of the tables the methods shown in the two following examples can be used, it being assumed as an approximation that the change in the function is proportional to the change in the angle, when the latter change is small.

Example 4.—Find tan 71° 20′.

From the table of tangents $\begin{cases} \tan 71° 18' = 2{\cdot}95437. \\ \tan 71° 24' = 2{\cdot}97144. \end{cases}$

Hence the difference for 6′ is 0·01707.

Therefore the difference for 2′ is $\frac{1}{3} \times 0{\cdot}01707 = 0{\cdot}00569$.

Thus, $\tan 71° 20' = 2{\cdot}95437 + 0{\cdot}00569 = 2{\cdot}96006$, approximately.

This process will seldom give accuracy to the fifth place of decimals. The value of tan 71° 20′, correct to five decimal places, is 2·96004.

Example 5.—Find from the tables the value of θ for which $\tan\theta = 3{\cdot}17545$.

$$\left.\begin{aligned} \tan 72° 30' &= 3{\cdot}17159 \\ \tan\theta &= 3{\cdot}17545 \end{aligned}\right\} \text{Difference} = 0{\cdot}00386.$$

Also $\tan 72° 36' = 3{\cdot}19100$; difference for $6' = 0{\cdot}01941$.

Hence
$$\theta \fallingdotseq 72° 30' + \frac{386}{1941} \times 6'$$
$$\fallingdotseq 72° 31'.$$

§ 7. Simple Trigonometric Equations

An equation involving circular functions of an unknown angle, which is not an identity, may become true when a definite value, or any one of a set of values is assigned to the angle. Such a value is a *root* of the equation, and the roots of the equation together form its *solution*.

At this stage it is instructive to consider a few trigonometric equations of simple type, which may be solved without the use of any formula other than possibly the fundamental identity.

The following standard results follow very simply from the definitions of the circular functions. Here, and in similar cases, n represents zero or any integer, positive or negative:

If $\cos\theta = 0$
or if $\cot\theta = 0$, $\quad \theta = \frac{1}{2}\pi + n\pi$. . . (4)

If $\sin\theta = 0$
or if $\tan\theta = 0$, $\quad \theta = n\pi$. . . (5)

$$\left.\begin{array}{lll} \text{If} & \cos\theta = 1, & \theta = 2n\pi, \\ \text{if} & \cos\theta = -1, & \theta = \pi + 2n\pi, \\ \text{if} & \sin\theta = 1, & \theta = \frac{1}{2}\pi + 2n\pi, \\ \text{if} & \sin\theta = -1, & \theta = \frac{3}{2}\pi + 2n\pi. \end{array}\right\} \quad . \quad (6)$$

Example 1.—Solve the equation $2 \operatorname{cosec}\theta = 3 + 2\sin\theta$.

The equation, expressed entirely in terms of $\sin\theta$, gives

$$\frac{2}{\sin\theta} = 3 + 2\sin\theta,$$

or

$$2 = 3\sin\theta + 2\sin^2\theta,$$

provided that

$$\sin\theta \neq 0.$$

This quadratic equation in $\sin\theta$ reduces to

$$(2\sin\theta - 1) \times (\sin\theta + 2) = 0,$$

which gives

$$\sin\theta = \tfrac{1}{2}$$

or $\quad \sin\theta = -2.$

Since $|\sin\theta| \leq 1$, no value of θ can be found for which $\sin\theta = -2$.

Taking $\sin\theta = \frac{1}{2}$, draw a rough sketch (Fig. 7) showing the possible directions of OP if $\angle XOP = \theta$.

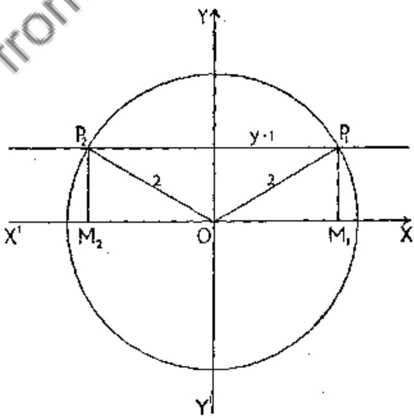

Fig. 7.

By definition $\sin\theta = \frac{y}{r}$, where y is the ordinate of P, and OP has length r. Therefore, in this case, $\frac{y}{r} = \frac{1}{2}$. Choose $r = 2$; then $y = 1$, and P must therefore be one of the points, P_1 and P_2, in which the line $y = 1$ meets the circle with centre O and radius 2; $\angle XOP_1$ and $\angle XOP_2$ thus satisfy the given equation.

Now, if M_1P_1 and M_2P_2 are the ordinates of P_1 and P_2, the angles at O in the triangles OM_1P_1, OP_2M_2 are 30°. Hence, $\angle XOP_1 = 30° + n \,.\, 360°$, $\angle XOP_2 = 150° + n \,.\, 360°$, where $n = 0, \pm 1, \pm 2, \ldots$; and the solution of the equation is $\theta = 30° + n \,.\, 360°$ or $150° + n \,.\, 360°$, none of these angles violating the restriction $\sin\theta \neq 0$.

Example 2.—Solve the equation

$$2\sin^2\theta + 5\cos\theta + 1 = 0.$$

Since by (1) $\sin^2\theta = 1 - \cos^2\theta$ for all values of θ, the given equation can be expressed as a quadratic equation in $\cos\theta$, namely

$$2\cos^2\theta - 5\cos\theta - 3 = 0.$$

This gives $\cos\theta = 3$ or $\cos\theta = -\frac{1}{2}$.

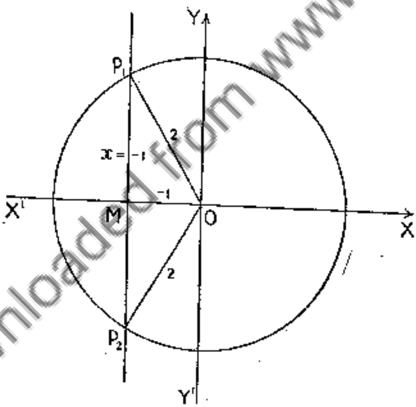

Fig. 8.

Since $|\cos\theta| \leqq 1$, no value of θ can be found such that $\cos\theta = 3$.

Taking $\cos\theta = -\frac{1}{2}$, and proceeding as in *Example* 1, we get $\frac{x}{r} = -\frac{1}{2}$. Let $r = 2$; then $x = -1$, and the solution of the given equation consists of $\angle XOP_1$ and $\angle XOP_2$ (Fig. 8), where P_1, P_2 are the points in which the line $x = -1$ meets the circle $x^2 + y^2 = 4$.

If P_1P_2 cuts the x-axis in M, the angles at O in the triangles OP_1M, OMP_2 are 60°. Therefore the complete solution of the given equation is

$$\theta = 120° + n \,.\, 360° \quad \text{or} \quad 240° + n \,.\, 360°,$$

where $n = 0, \pm 1, \pm 2, \ldots$.

Example 3.—Solve the equation $5\cos^2\theta + \sin\theta\cos\theta = 2$.

This equation may be expressed as a quadratic equation in $\tan\theta$ by dividing both sides by $\cos^2\theta$ and applying (2), or by writing the right-hand side as $2\cos^2\theta + 2\sin^2\theta$, which by (1) is equal to 2 for every value of θ, and then dividing by $\cos^2\theta$. The first process gives, if $\cos\theta \neq 0$,

$$\begin{aligned} 5 + \tan\theta &= 2\sec^2\theta \\ &= 2(1+\tan^2\theta), \end{aligned}$$

or
$$2\tan^2\theta - \tan\theta - 3 = 0,$$

whence
$$\tan\theta = -1 \quad \text{or} \quad \tan\theta = \tfrac{3}{2}.$$

Draw rough diagrams (Figs. 9, 10):

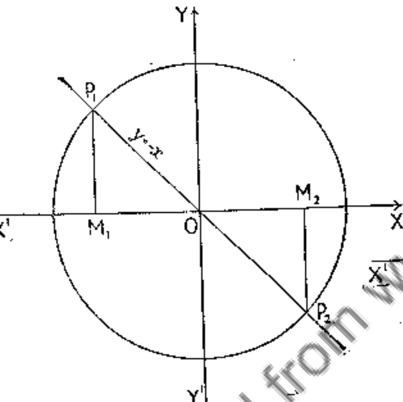

FIG. 9.

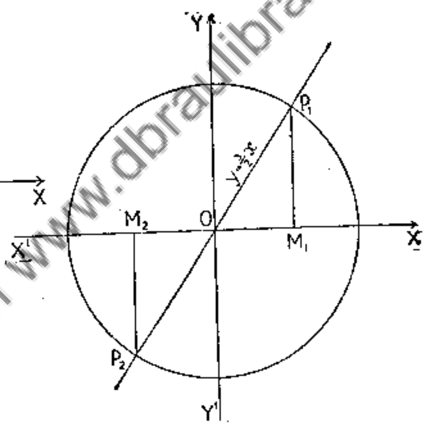

FIG. 10.

(i) $\tan\theta = -1 = \dfrac{y}{x}$. If P_1 and P_2 (Fig. 9) are the points in which the line $y = -x$ meets any circle with centre O and radius r, then $\angle XOP_1$ and $\angle XOP_2$ satisfy the equation. In the diagram the angles P_1OM_1 and P_2OM_2 are 45°; hence $\theta = 135° + n \,.\, 360°$, or $\theta = 315° + n \,.\, 360°$. Combining these into one expression, we have $\theta = 135° + n \,.\, 180°$, $n = 0, \pm 1, \pm 2, \ldots$.

(ii) $\tan\theta = \dfrac{3}{2} = \dfrac{y}{x}$. In this case P_1P_2 (Fig. 10) has equation $y = \dfrac{3}{2}x$. The positive acute angle XOP_1 is found from the table of tangents to be $56° 18\frac{1}{2}'$, correct to the nearest half-minute. Hence $\theta = 56° 18\frac{1}{2}' + n \,.\, 180°$, $n = 0, \pm 1, \pm 2, \ldots$

Thus, the complete solution of the given equation is

$$\theta = 135° + n \,.\, 180° \quad \text{or} \quad 56° 18\tfrac{1}{2}' + n \,.\, 180°,$$

where $n = 0, \pm 1, \pm 2, \ldots$ None of these angles violates the restriction $\cos\theta \neq 0$.

From the above examples it will be evident that the process of solution consists in first reducing the given equation to one or more of the simple types in which the value of a circular function is given explicitly, and then finding by means of a rough diagram, using tables where necessary, the angles which satisfy these simple equations. At a later stage the process can be condensed considerably (Ch. III, § 3). The student should at this stage solve some of the equations in Examples III, 9-26.

EXAMPLES II

1. Verify the following results in which c, s, t represent $\cos\theta$, $\sin\theta$ and $\tan\theta$, respectively:

$$\text{(i)}\ \cos\theta = \pm\sqrt{(1 - s^2)} = \pm 1/\sqrt{(1 + t^2)};$$
$$\text{(ii)}\ \sin\theta = \pm\sqrt{(1 - c^2)} = \pm t/\sqrt{(1 + t^2)};$$
$$\text{(iii)}\ \tan\theta = \pm\sqrt{(1 - c^2)}/c = \pm s/\sqrt{(1 - s^2)}.$$

2. If $\sin\theta = -0{\cdot}7$, find the values of $\cos\theta$ and $\tan\theta$ when θ is in (i) the third, (ii) the fourth quadrant.

Ans. (i) $\cos\theta = -\sqrt{(51)}/10$; $\tan\theta = 7/\sqrt{(51)}$;
(ii) $\cos\theta = \sqrt{(51)}/10$; $\tan\theta = -7/\sqrt{(51)}$.

3. If $\cos\theta = -3/5$, find the values of $\sin\theta$ and $\tan\theta$ when θ is in (i) the second, (ii) the third quadrant.

Ans. (i) $\sin\theta = 4/5$; $\tan\theta = -4/3$;
(ii) $\sin\theta = -4/5$; $\tan\theta = 4/3$.

4. If $\tan\theta = \sqrt{2}$, find the values of $\cos\theta$ and $\sin\theta$ when θ is in (i) the first, (ii) the third quadrant.

Ans. (i) $\cos\theta = \sqrt{(3)}/3$; $\sin\theta = \sqrt{(6)}/3$;
(ii) $\cos\theta = -\sqrt{(3)}/3$; $\sin\theta = -\sqrt{(6)}/3$.

5. If $\tan\theta = -2$, find the values of $\cos\theta$ and $\sin\theta$ when θ is in (i) the second, (ii) the fourth quadrant.

Ans. (i) $\cos\theta = -\sqrt{(5)}/5$; $\sin\theta = 2\sqrt{(5)}/5$;
(ii) $\cos\theta = \sqrt{(5)}/5$; $\sin\theta = -2\sqrt{(5)}/5$.

6. If $\sec\theta = 3$, find the values of $\sin\theta$ and $\tan\theta$ when θ is in (i) the first, (ii) the fourth quadrant.

Ans. (i) $\sin\theta = 2\sqrt{(2)}/3$; $\tan\theta = 2\sqrt{(2)}$;
(ii) $\sin\theta = -2\sqrt{(2)}/3$; $\tan\theta = -2\sqrt{(2)}$.

7. If $\operatorname{cosec}\theta = -2{\cdot}5$, find the values of $\cos\theta$ and $\tan\theta$ when θ is in (i) the third, (ii) the fourth quadrant.

Ans. (i) $\cos\theta = -\sqrt{(21)}/5$; $\tan\theta = 2\sqrt{(21)}/21$;
(ii) $\cos\theta = \sqrt{(21)}/5$; $\tan\theta = -2\sqrt{(21)}/21$.

8. If x is the smallest positive angle which satisfies the equation $12\tan x + 5 = 0$, calculate $\cos x$ and $\sin x$.

Ans. $\cos x = -12/13$; $\sin x = 5/13$.

9. The angle A of a triangle ABC is given by the equation $3\cos A + 2 = 0$. Find the values of $\sin A$ and $\tan A$.

Ans. $\sin A = \sqrt{(5)}/3$; $\tan A = -\sqrt{(5)}/2$.

10. Find the values of $\cos\theta$ and $\sin\theta$ when $\tan\theta = \dfrac{b}{a}$.

Ans. $\cos\theta = \pm\dfrac{a}{\sqrt{(a^2+b^2)}}$; $\sin\theta = \pm\dfrac{b}{\sqrt{(a^2+b^2)}}$.

11. Find the values of $\sin\theta$ and $\tan\theta$ when $\cos\theta = \dfrac{p}{q}$.

Ans. $\sin\theta = \pm\sqrt{(q^2-p^2)}/q$; $\tan\theta = \pm\sqrt{(q^2-p^2)}/p$.

12. Find the values of $\cos\theta$ and $\tan\theta$ when $\sin\theta = \dfrac{m}{n}$.

Ans. $\cos\theta = \pm\sqrt{(n^2-m^2)}/n$; $\tan\theta = \pm m/\sqrt{(n^2-m^2)}$.

13. Find the values of $\cos\theta$ and $\sin\theta$ when $\tan\theta = \dfrac{2n(n+1)}{2n+1}$.

Ans. $\cos\theta = \pm\dfrac{2n+1}{2n^2+2n+1}$; $\sin\theta = \pm\dfrac{2n(n+1)}{2n^2+2n+1}$.

14. Find the values of $\cos\theta$ and $\tan\theta$ when $\sin\theta = \dfrac{m^2-n^2}{m^2+n^2}$.

Ans. $\cos\theta = \pm\dfrac{2mn}{m^2+n^2}$; $\tan\theta = \pm\dfrac{m^2-n^2}{2mn}$.

15. If $0 < \theta < \frac{1}{2}\pi$ and $\tan^3\theta = b/a$, where a is (i) positive, (ii) negative, find the values of $a\sec\theta + b\operatorname{cosec}\theta$.

Ans. (i) $(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$; (ii) $-(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$

16. Using squared paper and a ruler, construct the angles in the following cases. Measure the angles with a protractor, and check your construction by reading their values from tables:

(i) The angles whose cosines have the value $\frac{5}{13}$;
(ii) the angles whose sines have the value $-\frac{8}{17}$;
(iii) the angles whose tangents have the value $-\frac{20}{21}$;
(iv) the angles whose secants have the value $-\frac{41}{9}$;
(v) the angles whose cosecants have the value $\frac{37}{35}$;
(vi) the angles whose cotangents have the value $\frac{60}{11}$.

Ans. (i) $67° 23'$; $292° 37'$;
(ii) $208° 4'$; $331° 56'$;
(iii) $136° 24'$; $316° 24'$;
(iv) $102° 41'$; $257° 19'$;
(v) $71° 5'$; $108° 55'$;
(vi) $10° 23'$; $190° 23'$.

17. Find from the tables the values of

(i) $\cos 5° 35'$; $\cos 74° 26'$; $\cos 87° 47'$;
(ii) $\sin 17° 52'$; $\sin 44° 16'$; $\sin 84° 31'$;
(iii) $\tan 10° 10'$; $\tan 45° 7'$; $\tan 79° 28'$;
(iv) $\sec 22° 13'$; $\sec 51° 45'$; $\sec 81° 19'$;
(v) $\operatorname{cosec} 25° 14'$; $\operatorname{cosec} 58° 44'$; $\operatorname{cosec} 82° 35'$;
(vi) $\cot 29° 39'$; $\cot 48° 37'$; $\cot 77° 53'$.

* Ans. (i) 0·99526; 0·26836; 0·03868;
(ii) 0·30680; 0·69800; 0·99542;
(iii) 0·17933; 1·00408; 5·37805;
(iv) 1·08019; 1·61526; 6·62369;
(v) 2·34573; 1·16992; 1·00844;
(vi) 1·75675; 0·88110; 0·21469.

18. Find from the tables, correct to the nearest half-minute, the values of the angle θ, between 0 and 90°, in the following cases:

(i) $\cos\theta = 0·61868$; (ii) $\sin\theta = 0·49354$;
(iii) $\tan\theta = 0·75240$; (iv) $\sec\theta = 2·07486$;
(v) $\operatorname{cosec}\theta = 1·32959$; (vi) $\cot\theta = 1·21643$.

Ans. (i) $51° 47'$; (ii) $29° 34\frac{1}{2}'$; (iii) $36° 57\frac{1}{2}'$;
(iv) $61° 11'$; (v) $48° 46\frac{1}{2}'$; (vi) $39° 25\frac{1}{2}'$.

Establish the identities in *Examples* 19–50:

19. $\cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$.

20. $\cos^2 A - \cos^4 A = \sin^2 A - \sin^4 A$.

21. $\cot\theta - \tan\theta = \sec\theta \operatorname{cosec}\theta (1 - 2\sin^2\theta)$.

22. $\tan^2\theta - \sin^2\theta = \tan^2\theta \sin^2\theta$.

23. $\cot^2\theta - \cos^2\theta = \cot^2\theta \cos^2\theta$.

* These values, correct to five decimals, are given for comparison with those obtained from the tables.

24. $\sec^2\theta + \operatorname{cosec}^2\theta = \sec^2\theta \operatorname{cosec}^2\theta$.

25. $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$.

26. $\dfrac{\tan\theta + \tan\phi}{\cot\theta + \cot\phi} = \tan\theta\tan\phi = \dfrac{\tan\theta - \tan\phi}{\cot\phi - \cot\theta}$.

27. $\dfrac{\tan\theta + \cot\phi}{\tan\phi + \cot\theta} = \tan\theta\cot\phi = \dfrac{\tan\theta - \cot\phi}{\tan\phi - \cot\theta}$.

28. $\sec x + \tan x = \dfrac{1 + \sin x}{\cos x} = \dfrac{\cos x}{1 - \sin x}$.

29. $\operatorname{cosec} x + \cot x = \dfrac{1 + \cos x}{\sin x} = \dfrac{\sin x}{1 - \cos x}$.

30. $\sec\theta + \tan\theta = \dfrac{1}{\sec\theta - \tan\theta}$.

31. $\operatorname{cosec}\theta + \cot\theta = \dfrac{1}{\operatorname{cosec}\theta - \cot\theta}$.

32. $\dfrac{1 - \sin\theta}{1 + \sin\theta} = (\sec\theta - \tan\theta)^2$.

33. $\dfrac{1 - \cos\theta}{1 + \cos\theta} = (\operatorname{cosec}\theta - \cot\theta)^2$.

34. $\dfrac{\sin x}{1 - \cos x} = \dfrac{1 + \cos x + \sin x}{1 - \cos x + \sin x}$.

35. $\dfrac{\cos x}{1 - \sin x} = \dfrac{1 + \cos x + \sin x}{1 + \cos x - \sin x}$.

36. $(\sec\phi - \cos\phi)(\operatorname{cosec}\phi - \sin\phi) = \cos\phi\sin\phi$.

37. $2\sec\phi\tan\phi = \dfrac{1}{\operatorname{cosec}\phi - 1} + \dfrac{1}{\operatorname{cosec}\phi + 1}$.

38. $2\tan^2\phi = \dfrac{1}{\operatorname{cosec}\phi - 1} - \dfrac{1}{\operatorname{cosec}\phi + 1}$.

39. $\cos^3\theta + \sin^3\theta = (\cos\theta + \sin\theta)(1 - \cos\theta\sin\theta)$.

40. $\cos^3\theta - \sin^3\theta = (\cos\theta - \sin\theta)(1 + \cos\theta\sin\theta)$.

41. $\cos^4\theta + \sin^4\theta = 1 - 2\cos^2\theta\sin^2\theta$.

42. $\cos^5\theta + \sin^5\theta$
$= (\cos\theta + \sin\theta)(1 - \cos\theta\sin\theta - \cos^2\theta\sin^2\theta)$.

43. $\cos^5\theta - \sin^5\theta$
$= (\cos\theta - \sin\theta)(1 + \cos\theta\sin\theta - \cos^2\theta\sin^2\theta)$.

44. $\cos^6\theta + \sin^6\theta = 1 - 3\cos^2\theta\sin^2\theta$.

45. $\cos^6\theta - \sin^6\theta = (\cos^2\theta - \sin^2\theta)(1 - \cos^2\theta\sin^2\theta)$.

46. $\operatorname{cosec}^2 A\cot^2 A - \sec^2 A\tan^2 A$
$= (\cot^2 A - \tan^2 A)(\sec^2 A\operatorname{cosec}^2 A - 1)$.

47. $\cos A(1 + \cot A) + \sin A(1 + \tan A)$
$$= \sec A + \operatorname{cosec} A.$$

48. $\cos^2 A \cos^2 B - \sin^2 A \sin^2 B = \cos^2 A - \sin^2 B.$

49. $\sin^2 A \cos^2 B - \cos^2 A \sin^2 B = \sin^2 A - \sin^2 B.$

50. $(\sec A \sec B + \tan A \tan B)^2$
$$- (\sec A \tan B + \tan A \sec B)^2 = 1.$$

51. Eliminate θ between the equations

$$x \cos \theta + y \sin \theta = 2,$$
$$y \cos \theta - x \sin \theta = 3.$$

Ans. $x^2 + y^2 = 13.$

52. Eliminate ϕ between the equations

$$x = 3 \cos \phi - 2 \sin \phi,$$
$$y = 4 \cos \phi + 3 \sin \phi.$$

Ans. $25x^2 - 12xy + 13y^2 = 289.$

53. Eliminate θ between the equations

$$a \cos \theta + b \sin \theta + c = 0,$$
$$a' \cos \theta + b' \sin \theta + c' = 0.$$

Ans. $(bc' - b'c)^2 + (ca' - c'a)^2 = (ab' - a'b)^2.$

54. If $a \cos \theta + b \sin \theta = c,$
and $b \cos \theta - a \sin \theta = d,$
show that $c^2 + d^2 = a^2 + b^2$; and deduce that the value of $a \cos \theta + b \sin \theta$ cannot be less than $-\sqrt{(a^2 + b^2)}$ or greater than $\sqrt{(a^2 + b^2)}$.

55. Eliminate α between the equations

$$x = \cos^2 \alpha - \sin^2 \alpha, \qquad y = 2 \cos \alpha \sin \alpha.$$

Ans. $x^2 + y^2 = 1.$

56. Eliminate β between the equations

$$x = \cos \beta\,(4 \cos^2\beta - 3), \qquad y = \sin \beta\,(4 \sin^2\beta - 3).$$

Ans. $x^2 + y^2 = 1.$

57. Eliminate θ between the equations

$$\cos \theta = n \sin \alpha, \qquad \cot \theta = \sin \alpha \cot \beta.$$

Ans. $n^2(\sec^2 \beta - \cos^2 \alpha) = 1.$

58. Eliminate β between the equations

$$x = \tan \beta + \sin \beta,$$
$$y = \tan \beta - \sin \beta.$$

Ans. $(x^2 - y^2)^2 = 16xy.$

59. Eliminate ϕ between the equations

$$x = \sec \phi - \tan \phi,$$
$$y = \operatorname{cosec} \phi + \cot \phi.$$

Ans. $xy + x - y + 1 = 0.$

60. Eliminate θ between the equations

$$\tan\theta - \cot\theta = a,$$
$$\cos\theta + \sin\theta = b.$$

Ans. $(a^2 + 4)(b^2 - 1)^2 = 4.$

61. If $x = a\cos^2\theta\sin\theta$ and $y = a\sin^2\theta\cos\theta$, show that $(x^2 + y^2)^3 = a^2x^2y^2$.

62. If $\cos x - \sin x = a$ and $\sec x + \operatorname{cosec} x = b$, show that $b^2 = (2 - a^2)(4 + a^2b^2)$.

63. If $a\sin x = b\cos x = \dfrac{2c\tan x}{1 - \tan^2 x}$, show that

$$(a^2 - b^2)^2 = 4c^2(a^2 + b^2).$$

64. Solve the equation $\cos 2\theta = 2\cos 70°$, giving the values of θ between 0 and 180°.

Ans. $23° 25'$; $156° 35'$.

65. Solve the equation $\tan 2\theta \tan 26° = 1$, giving the values of θ between 0 and 180°.

Ans. $32°$; $122°$.

66. Show that the equation $\cos x = a + \dfrac{1}{a}$, where a is real, has no real roots.

67. Show that the equation $\sin x = \dfrac{(a^2 + b^2)^2}{4a^2b^2}$, where a and b are real, has no real roots.

68. In a circle of radius a, P is a point on an arc AB which subtends a right angle at the centre O, and the tangents at A and B meet in T. Prove that, if $\angle AOP = \theta$,

$$TP^2 = a^2(3 - 2\cos\theta - 2\sin\theta),$$

and
$$\tan \widehat{ATP} = \frac{1 - \cos\theta}{1 - \sin\theta}.$$

69. ABCD is a square of side 12 inches. The circle with centre A and radius 13 inches cuts BC in E and CD in F. Calculate the area of the figure bounded by EC, CF and the arc FE.

Ans. 17·98 square inches.

70. A belt is stretched tightly round two circular discs which are in the same plane. If the radii of the discs are 3 feet and 4 feet, and their centres are 10 feet apart, calculate the length of the belt and the area enclosed by it.

Ans. 42·09 feet; 109·6 square feet.

71. Two wheels (radii R, r, where $R > r$) in the same plane are directly connected by a tight-fitting belt of length 2S. Show that

$$S = \pi R + (R - r)(\tan\theta - \theta),$$

where θ is the radian measure of the complement of the inclination of the straight part of the belt to the line of centres.

72. If in *Example* 71 the wheels have radii 6 feet and 1 foot, and their centres are 13 feet apart, calculate the length of the belt.

Calculate also the area, external to both wheels, contained by the two straight parts of the belt and the two wheels.

Ans. 49·94 feet; 39·70 square feet.

73. Two circles of radii r_1, r_2 $(r_1 > r_2)$ touch externally at A, and B, C are the points of contact of one of the direct common tangents. Show that BC $= 2\sqrt{(r_1 r_2)}$, and that the sine of the inclination of BC to the line of centres is $(r_1 - r_2)/(r_1 + r_2)$.

If $r_1 = 12$, $r_2 = 4$, find the area bounded by the smaller arcs AB, AC and the tangent BC.

Ans. 18·70 square units.

74. Prove that, if n is a positive integer,

$$\cos^{2n}\theta + \sin^{2n}\theta = 1 - n\sin^2\theta\cos^2\theta + \frac{n(n-3)}{2!}\sin^4\theta\cos^4\theta - \frac{n(n-4)(n-5)}{3!}\sin^6\theta\cos^6\theta + \dots,$$

the last term being $(-1)^{\frac{1}{2}n}\, 2\sin^n\theta\cos^n\theta$ if n is even, and $(-1)^{\frac{1}{2}n-\frac{1}{2}} n\sin^{n-1}\theta\cos^{n-1}\theta$ if n is odd.

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CHAPTER III

CIRCULAR FUNCTIONS OF RELATED ANGLES; EQUATIONS

§ 1. Relations connecting Circular Functions of certain related Angles

FORMULÆ will now be established by means of which the circular functions of the angles $\frac{1}{2}n\pi + \theta$ and $\frac{1}{2}n\pi - \theta$, where n is zero or an integer, can be expressed as circular functions of θ, with either the plus or the minus sign. The fundamental formulæ, from which the others can be deduced, are numbered I and II in the following discussion. The proofs which are given are perfectly general, and establish the formulæ for all values of θ.

I $\cos(-\theta) = \cos\theta$; $\sec(-\theta) = \sec\theta$;
$\sin(-\theta) = -\sin\theta$; $\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$;
$\tan(-\theta) = -\tan\theta$; $\cot(-\theta) = -\cot\theta$.

Let two radius vectors OP, OQ (Fig. 1), of the same length r, start from the same position along OX and describe angles which are equal in size but opposite in sign. Then OP and OQ, and consequently P and Q, are symmetrically placed with respect to the x-axis. Hence

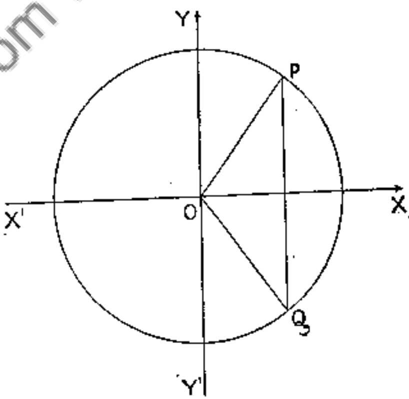


FIG. 1.

$$x_Q = x_P \quad \text{and} \quad y_Q = -y_P.$$

Now let $\angle XOP = \theta$, so that $\angle XOQ = -\theta$. Then, from the definitions of the cosine and the sine,

$$\cos(-\theta) = \frac{x_Q}{r} = \frac{x_P}{r} = \cos\theta,$$

and
$$\sin(-\theta) = \frac{y_Q}{r} = -\frac{y_P}{r} = -\sin\theta.$$

The other four results may either be established in a similar manner, or deduced from these two.

II $\cos(\frac{1}{2}\pi + \theta) = -\sin\theta$; $\sec(\frac{1}{2}\pi + \theta) = -\operatorname{cosec}\theta$;
$\sin(\frac{1}{2}\pi + \theta) = \cos\theta$; $\operatorname{cosec}(\frac{1}{2}\pi + \theta) = \sec\theta$;
$\tan(\frac{1}{2}\pi + \theta) = -\cot\theta$; $\cot(\frac{1}{2}\pi + \theta) = -\tan\theta$.

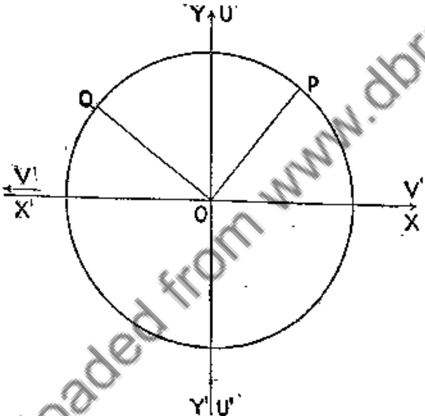


FIG. 2.

Let two radius vectors OP, OQ (Fig. 2), of the same length r, start from positions along OX, OY, respectively, and describe equal angles. Consider a new set of co-ordinate axes, with the same origin O, such that the axis of abscissæ U′OU coincides with Y′OY. Then the axis of ordinates V′OV, making a positive right angle with U′OU, lies along the x-axis but has the opposite direction positive. Hence the co-ordinates of Q with respect to the new and the original axes are connected by the equations

$$u_Q = y_Q \text{ and } v_Q = -x_Q.$$

Now the position of Q with respect to the new axes is the same as the position of P with respect to the original axes. Therefore

$$u_Q = x_P \text{ and } v_Q = y_P.$$

Hence $$y_Q = x_P \text{ and } x_Q = -y_P.$$

Let $\angle XOP = \theta$; then $\angle YOQ = \theta$, and

$$\angle XOQ = \angle XOY + \angle YOQ = \tfrac{1}{2}\pi + \theta.$$

Therefore, from the definitions of the cosine and the sine,

$$\cos(\tfrac{1}{2}\pi + \theta) = \frac{x_Q}{r} = -\frac{y_P}{r} = -\sin\theta,$$

and $$\sin(\tfrac{1}{2}\pi + \theta) = \frac{y_Q}{r} = \frac{x_P}{r} = \cos\theta.$$

The other four results follow in a similar manner, or may be deduced from these two.

III $\cos(\tfrac{1}{2}\pi - \theta) = \sin\theta$; $\sec(\tfrac{1}{2}\pi - \theta) = \operatorname{cosec}\theta$;
$\sin(\tfrac{1}{2}\pi - \theta) = \cos\theta$; $\operatorname{cosec}(\tfrac{1}{2}\pi - \theta) = \sec\theta$;
$\tan(\tfrac{1}{2}\pi - \theta) = \cot\theta$; $\cot(\tfrac{1}{2}\pi - \theta) = \tan\theta$.

These results follow at once from II and I. For example,

$$\begin{aligned}\cos(\tfrac{1}{2}\pi - \theta) &= \cos\{\tfrac{1}{2}\pi + (-\theta)\}\\ &= -\sin(-\theta), \text{ by II},\\ &= \sin\theta, \text{ by I};\\ \sin(\tfrac{1}{2}\pi - \theta) &= \sin\{\tfrac{1}{2}\pi + (-\theta)\}\\ &= \cos(-\theta), \text{ by II},\\ &= \cos\theta, \text{ by I}.\end{aligned}$$

IV $\cos(\pi + \theta) = -\cos\theta$; $\sec(\pi + \theta) = -\sec\theta$;
$\sin(\pi + \theta) = -\sin\theta$; $\operatorname{cosec}(\pi + \theta) = -\operatorname{cosec}\theta$
$\tan(\pi + \theta) = \tan\theta$; $\cot(\pi + \theta) = \cot\theta$.

These formulæ can be proved by a double application of II.

For example,

$$\begin{aligned}\cos(\pi + \theta) &= \cos\{\tfrac{1}{2}\pi + (\tfrac{1}{2}\pi + \theta)\}\\ &= -\sin(\tfrac{1}{2}\pi + \theta), \text{ by II},\\ &= -\cos\theta, \text{ by II};\\ \sin(\pi + \theta) &= \sin\{\tfrac{1}{2}\pi + (\tfrac{1}{2}\pi + \theta)\}\\ &= \cos(\tfrac{1}{2}\pi + \theta), \text{ by II},\\ &= -\sin\theta, \text{ by II}.\end{aligned}$$

Period of tan θ *and* cot θ.—Since, for all values of θ, $\tan(\theta + \pi) = \tan\theta$, and $\cot(\theta + \pi) = \cot\theta$, the functions $\tan\theta$ and $\cot\theta$ have the period π radians or 180°.

V $\cos(\pi - \theta) = -\cos\theta$; $\sec(\pi - \theta) = -\sec\theta$;
$\sin(\pi - \theta) = \sin\theta$; $\operatorname{cosec}(\pi - \theta) = \operatorname{cosec}\theta$;
$\tan(\pi - \theta) = -\tan\theta$; $\cot(\pi - \theta) = -\cot\theta$.

To establish these results, either II and III or IV and I may be applied.

For example,

$$\begin{aligned} \cos(\pi - \theta) &= \cos\{\tfrac{1}{2}\pi + (\tfrac{1}{2}\pi - \theta)\} \\ &= -\sin(\tfrac{1}{2}\pi - \theta), \text{ by II}, \\ &= -\cos\theta, \text{ by III}; \\ \sin(\pi - \theta) &= \sin\{\pi + (-\theta)\} \\ &= -\sin(-\theta), \text{ by IV}, \\ &= \sin\theta, \text{ by I}. \end{aligned}$$

The student should make himself thoroughly familiar with formulæ I to V.

Example 1.—Prove that (i) $\cos(270° - A) = -\sin A$;
(ii) $\sin(270° + A) = -\cos A$.

$$\begin{aligned} \text{(i) } \cos(270° - A) &= \cos\{180° + (90° - A)\} \\ &= -\cos(90° - A), \text{ by IV}, \\ &= -\sin A, \text{ by III}. \\ \text{(ii) } \sin(270° + A) &= \sin\{180° + (90° + A)\} \\ &= -\sin(90° + A), \text{ by IV}, \\ &= -\cos A, \text{ by II}. \end{aligned}$$

Example 2.—Show that (i) $\sin(540° - \theta) = \sin\theta$;
(ii) $\sec(630° - \theta) = -\operatorname{cosec}\theta$.

$$\begin{aligned} \text{(i) } 540° - \theta &= 360° + (180° - \theta). \\ \text{Hence } \sin(540° - \theta) &= \sin(180° - \theta) = \sin\theta, \text{ by V}. \\ \text{(ii) } 630° - \theta &= 2 \times 360° - (90° + \theta). \\ \text{Hence } \sec(630° - \theta) &= \sec\{-(90° + \theta)\} \\ &= \sec(90° + \theta), \text{ by I}, \\ &= -\operatorname{cosec}\theta, \text{ by II}. \end{aligned}$$

The student may verify the following useful working rules for expressing any given circular function of an angle of the form $(\tfrac{1}{2}n\pi \pm \theta)$ in terms of a circular function of θ. Since the form of the result is the same for all values of θ, that form may legitimately be determined by assuming

θ to be in the first quadrant, and therefore each of the circular functions of θ to be positive.

(i) Assuming that $0 < \theta < 90°$, note the quadrant in which the given angle lies. The result has the plus or the minus sign according as the given function is positive or negative in that quadrant.

(ii) If n is *even*, the result contains the same circular function as the given expression; but, if n is *odd*, the result contains the corresponding *co-function*, that is, sine becomes cosine, tangent becomes cotangent, secant becomes cosecant, and vice versa.

For instance, in *Example* 2 above,

(i) $540° - \theta = 6 \times 90° - \theta$, a *second* quadrant angle if $0 < \theta < 90°$; in this quadrant the sine is *positive*. Since the angle contains an *even* multiple of 90°, the sine is retained. Hence $\sin (540° - \theta) = + \sin \theta$.

(ii) $630° - \theta = 7 \times 90° - \theta$, a *third* quadrant angle if $0 < \theta < 90°$; in this quadrant the secant is *negative*. Since the angle contains an *odd* multiple of 90°, the secant is replaced by cosecant. Hence $\sec (630° - \theta) = - \operatorname{cosec} \theta$.

Since it is possible to express any specified angle in one or other of the forms $(n \,.\, 90° \pm \alpha)$, where n is zero or an integer, and $0 \leqq \alpha \leqq 45°$, any circular function of any specified angle can be expressed as a circular function of an angle in the range from 0 to 45° inclusive, with the appropriate sign.

For example,

$$\cot (-769°) = \cot (-9 \times 90° + 41°) = -\tan 41°;$$
$$\cos 1027° = \cos (11 \times 90° + 37°) = \sin 37°.$$

Example 3.—Find from the tables the values of (i) $\tan 710°$; (ii) $\sin (-663°)$; (iii) $\operatorname{cosec} 949°$.

Ans. (i) $-0{\cdot}17633$; (ii) $0{\cdot}83867$; (iii) $-1{\cdot}32501$.

Example 4.—Without reference to tables, show that

$$\operatorname{cosec} 317° - \sec (-586°) < 0.$$

Since $\operatorname{cosec} 317° = \operatorname{cosec} (360° - 43°) = -\operatorname{cosec} 43°$,
and $\sec (-586°) = \sec (7 \times 90° - 44°) = -\operatorname{cosec} 44°$,

$$\operatorname{cosec} 317° - \sec (-586°) = \operatorname{cosec} 44° - \operatorname{cosec} 43°.$$

Now, for the range $0 < \theta < 90°$, $\operatorname{cosec} \theta$ decreases as θ increases; hence $\operatorname{cosec} 44° < \operatorname{cosec} 43°$, and therefore

$$\operatorname{cosec} 317° - \sec (-586°) < 0.$$

§ 2. General Expressions for Angles for which a Circular Function has a given Value

In Ch. II, § 7, examples were given of the determination of angles from a known value of one circular function. The three general cases, in which the value of the cosine, the sine or the tangent is given, will now be considered.

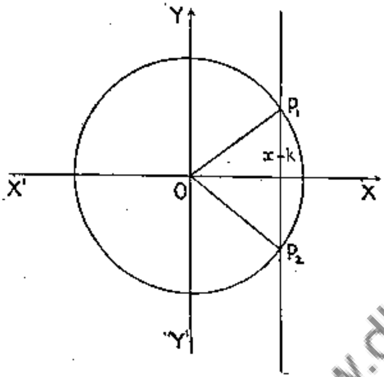


FIG. 3.

I. Given that $\cos\theta = k$, where

$$-1 \leqq k \leqq 1,$$

to find all the possible values of θ.

Let P_1, P_2 (Fig. 3) be the points in which the straight line $x = k$ meets the circle whose centre is O and whose radius is unity.

Then, by definition,

$$\cos X\hat{O}P_1 = \frac{x_{P_1}}{1} = k,$$

and
$$\cos X\hat{O}P_2 = \frac{x_{P_2}}{1} = k;$$

and, since P_1 and P_2 are the only points on the circle whose abscissæ have the value k, OP_1 and OP_2 are the only final positions of the radius vector which define angles whose cosines have the value k.

Now OP_1, OP_2 are symmetrically placed with respect to the x-axis, so that, if one value of $\angle XOP_1$ is α, one value of $\angle XOP_2$ is $-\alpha$. Hence all angles whose cosines have the value k are given by the expressions

$$\alpha + 2n\pi \text{ and } -\alpha + 2n\pi,$$

where α is any one such angle, and n is zero or an integer.

This result may be summarised as follows.

If $\cos\theta = \cos\alpha$, then $\theta = \pm\alpha + 2n\pi$, . (1)

where $n = 0, \pm 1, \pm 2, \ldots$

II. Given that $\sin\theta = k$, where $-1 \leqq k \leqq 1$, to find all the possible values of θ.

This case may be discussed from first principles, the line $y = k$ being drawn to cut the circle whose centre is O and whose radius is unity. This is left as an exercise to the student.

Alternatively, the result of Case I may be applied, thus.

Let α be any angle whose sine has the value k. Then $\sin\theta = \sin\alpha$, or

$$\cos\left(\theta - \frac{\pi}{2}\right) = \cos\left(\alpha - \frac{\pi}{2}\right).$$

Therefore, by (1),

$$\theta - \frac{\pi}{2} = \alpha - \frac{\pi}{2} + 2n\pi,$$

or

$$\theta - \frac{\pi}{2} = -\left(\alpha - \frac{\pi}{2}\right) + 2n\pi,$$

where $n = 0, \pm 1, \pm 2, \ldots$

Hence $\theta = \alpha + 2n\pi$ or $(\pi - \alpha) + 2n\pi$.

It has therefore been proved that

if $\sin\theta = \sin\alpha$, then $\left.\begin{array}{l}\theta = \alpha + 2n\pi \\ \theta = (\pi - \alpha) + 2n\pi\end{array}\right\}$. (2)

or

where $n = 0, \pm 1, \pm 2, \ldots$

Note.—The expressions in (2), being α plus an *even* multiple of π, and $-\alpha$ plus an *odd* multiple of π, can be combined into the single expression $(-1)^m\alpha + m\pi$, where m is zero or an integer. In practical work, however, it is better to retain the separate forms, remembering the set of angles as comprising α, $\pi - \alpha$, and all angles coterminal with either.

III. Given that $\tan\theta = k$, to find all the possible values of θ.

Let P_1, P_2 (Fig. 4) be the points in which the line $y = kx$ cuts a fixed circle whose centre is O.

Then, by definition,

$$\tan X\widehat{O}P_1 = \frac{y_{P_1}}{x_{P_1}} = k,$$

and
$$\tan X\widehat{O}P_2 = \frac{y_{P_2}}{x_{P_2}} = k.$$

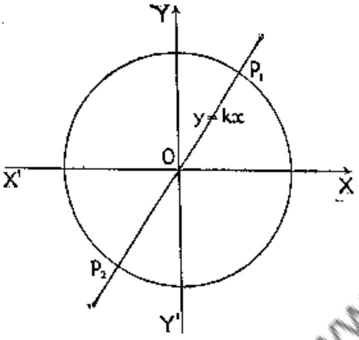


FIG. 4.

Further, there is no other point on the circle at which $y/x = k$, and therefore OP_1, OP_2 are the only final positions of the radius vector which define angles whose tangents have the value k. It follows that, if one value of $\angle XOP_1$ is α, all the angles whose tangents have the value k are given by the expression $\alpha + n\pi$,

where $\qquad n = 0, \pm 1, \pm 2, \ldots$

Therefore,

if $\qquad \tan\theta = \tan\alpha$, then $\theta = \alpha + n\pi, \quad . \quad . \quad (3)$

where $\qquad n = 0, \pm 1, \pm 2, \ldots$

§ 3. Solution of Equations

The standard formulæ (1), (2), (3) of § 2 enable the process of solving a trigonometric equation, as set out in the worked examples at the end of Chap. II, to be shortened considerably. For instance, in *Example* 1, p. 29, from the stage $\sin\theta = \frac{1}{2}$, the solution can be completed as follows:

$$\sin\theta = \tfrac{1}{2} = \sin 30^\circ;$$

therefore, by (2),

$$\theta = 30^\circ + n \,.\, 360^\circ \quad \text{or} \quad \theta = 150^\circ + n \,.\, 360^\circ,$$

where $\qquad n = 0, \pm 1, \pm 2, \ldots$

Example 1.—Solve the equation

$$2 \tan^2 x = 7 - 3 \sec x.$$

If $\tan^2 x$ is replaced by $\sec^2 x - 1$, the equation becomes

$$2 \sec^2 x + 3 \sec x - 9 = 0,$$

or, provided that $\cos x \neq 0$,

$$9 \cos^2 x - 3 \cos x - 2 = 0.$$

Hence $$(3 \cos x + 1)(3 \cos x - 2) = 0,$$
and therefore $$\cos x = -\tfrac{1}{3} \quad \text{or} \quad \tfrac{2}{3}.$$

If $\cos x = -\tfrac{1}{3} \fallingdotseq -0{\cdot}33333 \fallingdotseq -\cos 70° \, 32'$,
$\fallingdotseq \cos 109° \, 28'$,
then $x \fallingdotseq \pm 109° \, 28' + n \, . \, 360°$, where $n = 0, \pm 1, \pm 2, \ldots$.

If $\cos x = \tfrac{2}{3} \fallingdotseq 0{\cdot}66667 \fallingdotseq \cos 48° \, 11'$, then $x \fallingdotseq \pm 48° \, 11' + n \, . \, 360°$, where $n = 0, \pm 1, \pm 2, \ldots$.

The complete solution is thus

$x \fallingdotseq \pm 48° \, 11' + n \, . \, 360°$, or $x \fallingdotseq \pm 109° \, 28' + n \, . \, 360°$,

where $n = 0, \pm 1, \pm 2, \ldots$.

Should the solutions in any specified range be required, they can be obtained by assigning to n the appropriate values; for example, the solutions in the range from $x = 0$ to $x = 360°$ are $48° \, 11'$, $109° \, 28'$, $250° \, 32'$, $311° \, 49'$.

Example 2.—Solve the equation

$$3 \cos^2 \theta - \sin \theta = 2.$$

This equation reduces to

$$3 \sin^2 \theta + \sin \theta - 1 = 0,$$

whence $\sin \theta = \tfrac{1}{6}(-1 \pm \sqrt{13}) \fallingdotseq \tfrac{1}{6}(-1 \pm 3{\cdot}60555)$.

The minus sign gives

$$\sin \theta \fallingdotseq -0{\cdot}76759 \fallingdotseq -\sin 50° \, 8' = \sin(-50° \, 8'),$$

from which $\theta \fallingdotseq -50° \, 8'$ or $180° - (-50° \, 8')$, or any angle coterminal with either.

The plus sign gives

$$\sin \theta \fallingdotseq 0{\cdot}43426 \fallingdotseq \sin 25° \, 44',$$

from which $\theta \fallingdotseq 25° \, 44'$ or $180° - 25° \, 44'$, or any angle coterminal with either.

The complete solution therefore consists of the four sets of angles

$$25° \, 44' + n \, . \, 360°; \quad 154° \, 16' + n \, . \, 360°; \quad 230° \, 8' + n \, . \, 360°; \quad 309° \, 52' + n \, . \, 360°;$$

where $n = 0, \pm 1, \pm 2, \ldots$.

Example 3.—Solve the equation

$$3 + 4 \sin \theta \cos \theta = 5 \sin^2 \theta.$$

This equation gives, provided that $\cos \theta \neq 0$,

$$3 \sec^2 \theta + 4 \tan \theta = 5 \tan^2 \theta,$$

or $$2 \tan^2 \theta - 4 \tan \theta - 3 = 0.$$

Therefore $\tan\theta = \frac{1}{2}(2 \pm \sqrt{10}) \fallingdotseq 1 \pm \frac{1}{2} \times 3{\cdot}16228$;
so that $\tan\theta \fallingdotseq 2{\cdot}58114 \fallingdotseq \tan 68^\circ\, 49'$,
or $\tan\theta \fallingdotseq -0{\cdot}58114 \fallingdotseq -\tan 30^\circ\, 10' = \tan 149^\circ\, 50'$.

The complete solution is thus

$$\theta \fallingdotseq 68^\circ\, 49' + n\,.\,180^\circ \quad \text{or} \quad 149^\circ\, 50' + n\,.\,180^\circ,$$

where $n = 0, \pm 1, \pm 2, \ldots$.

Example 4.—Solve the equation $\tan x \tan 4x = 1$.

Provided that $\tan x$ is neither zero nor infinite, the equation reduces to

$$\tan 4x = \cot x = \tan(\tfrac{1}{2}\pi - x),$$

whence $4x = \frac{1}{2}\pi - x + n\pi$,

or $$x = (2n+1)\frac{\pi}{10}, \text{ where } n = 0, \pm 1, \pm 2, \ldots.$$

For every angle of this set, the condition $\tan x \neq 0$ is satisfied; but the condition $\tan x \neq \infty$ is violated if $\dfrac{2n+1}{5} = \pm 1, \pm 3, \pm 5, \ldots$, that is, if $n = -3, -8, -13, \ldots;\ 2, 7, 12, \ldots$. For each of the angles given by these values of n, the left side of the equation takes the indeterminate form $\infty \times 0$, and the equation is not satisfied.

Finally, the complete solution is

$$x = (2n+1)\frac{\pi}{10},$$

where n is zero or any integer excepting $\ldots, -13, -8$ $-3, 2, 7, 12, \ldots$, that is, excepting those of the form $5r + 2$, where r is zero or an integer.

EXAMPLES III

1. Express as circular functions of angles between 0 and 45° (i) $\operatorname{cosec}(-969^\circ)$; (ii) $\sin(-1484^\circ)$; (iii) $\cos 1027^\circ$; (iv) $\cot(769^\circ)$; (v) $\tan(-785^\circ)$; (vi) $\sec(-851^\circ)$.

Ans. (i) $\sec 21^\circ$; (ii) $-\sin 44^\circ$; (iii) $\sin 37^\circ$; (iv) $\tan 41^\circ$; (v) $-\cot 25^\circ$; (vi) $-\operatorname{cosec} 41^\circ$.

Using tables, find approximately the values of
sin $1314^\circ\, 26'$; (ii) $\operatorname{cosec} 855^\circ\, 9'$; (iii) $\tan 883^\circ\, 15'$;
os $575^\circ\, 33'$; (v) $\sec 1184^\circ\, 49'$; (vi) $\cot 251^\circ\, 39'$.

(i) $-0{\cdot}81344$; (ii) $1{\cdot}41793$; (iii) $-0{\cdot}30097$;
(iv) $-0{\cdot}81361$; (v) $-3{\cdot}91042$; (vi) $0{\cdot}33169$.

hout using tables, show that

(i) $\tan 672^\circ + \cot(-497^\circ) < 0$;
(ii) $\cos 577^\circ - \sin 677^\circ < 0$;
(iii) $\operatorname{cosec} 611^\circ + \sec(-741^\circ) > 0$;
(iv) $\sin(-651^\circ) - \cos(691^\circ) > 0$.

4. Find the values of θ between 0 and 360° which satisfy the equations :

(i) $\cos\theta = -\sin 325^\circ$; (ii) $\tan\theta = -\cot 500^\circ$;

(iii) $\sin\theta = \sin(-683^\circ)$; (iv) $\cos\theta = \cos(-474^\circ)$.

Ans. (i) $55^\circ, 305^\circ$; (ii) $50^\circ, 230^\circ$;

(iii) $37^\circ, 143^\circ$; (iv) $114^\circ, 246^\circ$.

Establish the identities in Examples 5-8 :

5. $\cos^2(90^\circ+\theta) + \operatorname{cosec}^2(180^\circ+\theta) + \sin^2(270^\circ-\theta)$
$= 2 + \tan^2(270^\circ+\theta)$.

6. $\{2 - \sec(\frac{1}{2}\pi+\theta)\}\{\cot(\frac{1}{2}\pi+\theta) - \sec(\pi-\theta)\}$
$= \tan(\frac{3}{2}\pi-\theta) + \tan(3\pi-\theta) - \operatorname{cosec}(\frac{3}{2}\pi+\theta)$.

7. $\{\operatorname{cosec}(\theta-3\pi)+\cos(\theta+\frac{5}{2}\pi)\}\{\sec(5\pi-\theta)-\sin(\theta-\frac{9}{2}\pi)\}$
$= \tan(\theta-\pi)\{1+\sin^2(\theta-3\pi)\}$.

8. $\operatorname{cosec}(\frac{3}{2}\pi-x)\{\sec(\frac{7}{2}\pi+x)+2\sin(3\pi+x)\}$
$= 2\tan(x-\frac{5}{2}\pi) + \sec(x+\frac{5}{2}\pi)\sec(x+5\pi)$.

Solve the equations in *Examples* 9-33, giving in each case the values of θ in the range from 0 to 360°.

9. $3\sin\theta + 2\tan\theta = 0$.
Ans. $0, 180^\circ, 360^\circ$; $131^\circ\ 48\frac{1}{2}'$, $228^\circ\ 11\frac{1}{2}'$.

10. $2\cos\theta = \cot\theta$.
Ans. $90^\circ, 270^\circ$; $30^\circ, 150^\circ$.

11. $2\cos\theta = 3\tan\theta$.
Ans. $30^\circ, 150^\circ$.

12. $3\tan^2\theta = 2\sin\theta$.
Ans. $0, 180^\circ, 360^\circ$; $30^\circ, 150^\circ$.

13. $6\tan\theta - 2\cot\theta = 1$.
Ans. $33^\circ\ 41\frac{1}{2}'$, $213^\circ\ 41\frac{1}{2}'$; $153^\circ\ 26'$, $333^\circ\ 26'$.

14. $3\tan\theta = \sec^2\theta + 1$.
Ans. $45^\circ, 225^\circ$; $63^\circ\ 26'$, $243^\circ\ 26'$.

15. $2\sin^2\theta = 5\cos\theta - 1$.
Ans. $60^\circ, 300^\circ$.

16. $6\cos\theta - 2\sec\theta = 1$.
Ans. $120^\circ, 240^\circ$; $48^\circ\ 11\frac{1}{2}'$, $311^\circ\ 48\frac{1}{2}'$.

17. $5\sin\theta = 7\cot\theta - \operatorname{cosec}\theta$.
Ans. $53^\circ\ 8'$, $306^\circ\ 52'$.

18. $\sin\theta = 1 - 2\cos^2\theta$.
Ans. 90° ; $210^\circ, 330^\circ$.

19. $6\sin\theta + 6\operatorname{cosec}\theta = 13$.
Ans. $41^\circ\ 48\frac{1}{2}'$, $138^\circ\ 11\frac{1}{2}'$.

20. $11\sec\theta + 3\tan\theta = 20\cos\theta$.
Ans. $36^\circ\ 52'$, $143^\circ\ 8'$; $228^\circ\ 35\frac{1}{2}'$, $311^\circ\ 24\frac{1}{2}'$.

21. $8\cos^4\theta + 8\sin^4\theta = 5$.
Ans. $30^\circ, 150^\circ, 210^\circ, 330^\circ$; $60^\circ, 120^\circ, 240^\circ, 300^\circ$.

22. $6 \sin^2 \theta - \sin \theta \cos \theta - 12 \cos^2 \theta = 0$.
Ans. $56° 18\frac{1}{2}'$, $236° 18\frac{1}{2}'$; $126° 52'$, $306° 52'$.

23. $11 \sin^2 \theta + 10 \sin \theta \cos \theta + 5 \cos^2 \theta = 3$.
Ans. $135°$, $315°$; $165° 58'$, $345° 58'$.

24. $8 \sin^2 \theta - \sin \theta \cos \theta = 6$.
Ans. $63° 26'$, $243° 26'$; $123° 41\frac{1}{2}'$, $303° 41\frac{1}{2}'$.

25. $11 \cos \theta - 13 \sin \theta = 5 \sec \theta$.
Ans. $21° 48'$, $201° 48'$; $108° 26'$, $288° 26'$.

26. $7 \sin \theta + 4 \cos \theta = 4 \operatorname{cosec} \theta$.
Ans. $33° 41\frac{1}{2}'$, $213° 41\frac{1}{2}'$; $116° 34'$, $296° 34'$.

27. $\tan 2x \tan 3x = 1$.
Ans. $18°$, $54°$, $126°$, $162°$, $198°$, $234°$, $306°$, $342°$.

28. $\tan (2x + 30°) + \tan x = 0$.
Ans. $50°$, $110°$, $170°$, $230°$, $290°$, $350°$.

29. $\tan (3x + 52°) + \tan x = 0$.
Ans. $32°$, $77°$, $122°$, $167°$, $212°$, $257°$, $302°$, $347°$.

30. $\cos 3x + \sin 2x = 0$.
Ans. $54°$, $90°$, $126°$, $198°$, $270°$, $342°$.

31. $\cos (2x + 63°) + \cos (x + 9°) = 0$.
Ans. $36°$, $126°$, $156°$, $276°$.

32. $\cos 3x = \sin 2x$.
Ans. $18°$, $90°$, $162°$, $234°$, $270°$, $306°$.

33. $\sin 7x + \sin (3x + 60°) = 0$.
Ans. $30°$, $60°$, $66°$, $102°$, $138°$, $150°$, $174°$, $210°$, $240°$, $246°$, $282°$, $318°$, $330°$, $354°$.

CHAPTER IV

GRAPHS; INVERSE FUNCTIONS; AREA OF A SEGMENT OF A CIRCLE

§ 1. Graphs of the Circular Functions

FROM tables, one or more of the formulæ proved in Chap. III, § 1, being employed if necessary, the values of the circular functions of any specified angle can be found. Graphs of the circular functions can therefore now be drawn for any range of values of the angle.

Throughout this section it will be understood, unless otherwise indicated, that angles are expressed in degrees. Thus $\cos x$ will be printed for $\cos x°$, the degree symbol being omitted for convenience.

Graph of cos x.—Table 1 shows, correct to two places of decimals, the values of $\cos x$ for values of x from 0 to 360° by steps of 20°. The zero values for $x = 90°$ and for $x = 270°$ are also shown.

TABLE 1

x	0	20°	40°	60°	80°	90°	100°	120°	140°	160°
$\cos x$	1	0·94	0·77	0·50	0·17	0	−0·17	−0·50	−0·77	−0·94

x	180°	200°	220°	240°	260°	270°	280°	300°	320°	340°	360°
$\cos x$	−1	−0·94	−0·77	−0·50	−0·17	0	0·17	0·50	0·77	0·94	1

The graph of $\cos x$, that is, the graph whose equation is $y = \cos x$, is shown in Fig. 1 for the range from $x = -360°$ to $x = 720°$.

Since $\cos (x + n \,.\, 360°) = \cos x$, n being an integer, the ordinates at the points whose abscissæ are x, $x \pm 360°$, $x \pm 2 \times 360°$, . . . are equal. Hence, if any point on the graph is moved 360° to the right or to the left, it coincides with another point on the graph; and therefore any part of the graph, if moved 360° right or left, coincides with another part. The complete curve thus consists of the wave from $x = 0$ to $x = 360°$, and an endless repetition of that wave to the left and to the right. This repetition is, of course, merely the graphical expression of the periodicity of cos x.

The curve has an infinite number of maximum turning points,* at intervals of 360° on the line $y = 1$, and an infinite number of minimum turning points,* at intervals of

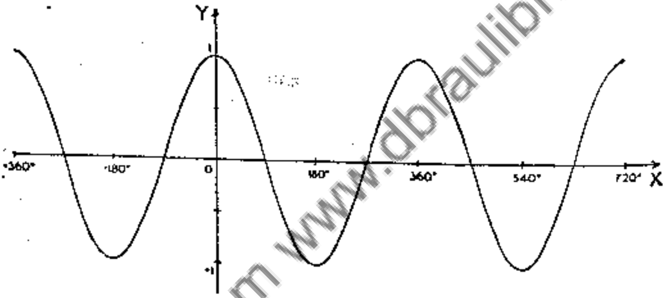


FIG. 1. $y = \cos x$.

360° on the line $y = -1$. All other points on the curve lie between $y = -1$ and $y = 1$.

Since $\cos(-x) = \cos x$, the graph of cos x is symmetrical about the y-axis.

Graph of sin x.—Fig. 2 shows the graph of sin x for the range from $x = -360°$ to $x = 720°$, drawn from the data contained in Table 2.

Since $\sin(x + 360°) = \sin x$, this graph, like the graph of cos x, consists of an endless succession of identical parts, each corresponding to a range of 360° on the x-axis.

* Let a curve be considered as traced out by a moving point. A point at which the ordinate of the moving point ceases to increase and begins to decrease is a *maximum turning point;* a point at which the ordinate ceases to decrease and begins to increase is a *minimum turning point.*

TABLE 2

x	0	20°	40°	60°	80°	90°	100°	120°	140°	160°	180°
$\sin x$	0	0·34	0·64	0·87	0·98	1	0·98	0·87	0·64	0·34	0

x	200°	220°	240°	260°	270°	280°	300°	320°	340°	360°
$\sin x$	−0·34	−0·64	−0·87	−0·98	−1	−0·98	−0·87	−0·64	−0·34	0

The graphs of $\cos x$ and $\sin x$ are identical in form, differing only in position. Since $\cos x = \sin(x + 90°)$, the ordinate of the cosine curve for any given abscissa is equal to the ordinate of the sine curve for abscissa 90° greater. Thus the graph of $\cos x$, if moved to the right through 90°, becomes the graph of $\sin x$.

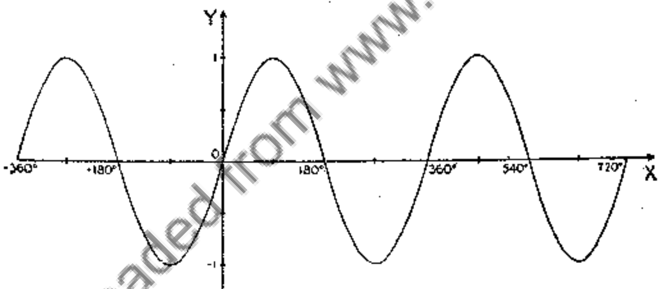


FIG. 2. $y = \sin x$.

Graph of tan x.—Since $\tan(x + 180°) = \tan x$, the graph of $\tan x$ consists of an infinite number of identical parts, each corresponding to a range of 180° on the x-axis. Table 3 shows the values of $\tan x$ for values of x from $-90°$ to 90° by steps of 20°, and the graph of $\tan x$, from $x = -270°$ to $x = 270°$, is shown in Fig. 3.

There are no points on the graph of $\tan x$ when $x = \pm 90°$, $\pm 270°$, $\pm 450°$, . . . The curve approaches more and more closely to the lines $x = \pm 90°$, $\pm 270°$, . . . when

TABLE 3.

x .	0	$\pm 10°$	$\pm 30°$	$\pm 50°$	$\pm 70°$	$\pm 90°$
tan x .	0	$\pm 0{\cdot}18$	$\pm 0{\cdot}58$	$\pm 1{\cdot}19$	$\pm 2{\cdot}75$	$\pm \infty$

x approaches these values. These lines are called asymptotes to the curve.

Throughout its whole course, the graph rises from left to right. This is the graphical expression of the fact

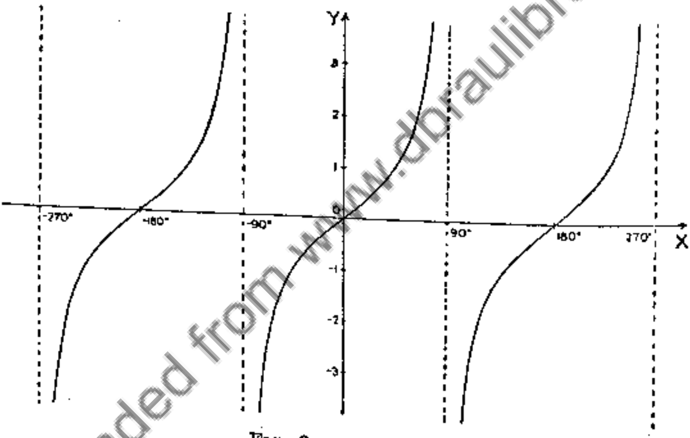


FIG. 3. $y = \tan x$.

that tan x always increases as x increases, except when x goes through a value for which tan x has no value.

Graphs of sec x, cosec x and cot x.—The student should draw the graphs of sec x and cosec x from $x = -360°$ to $x = 720°$ and note how their main features can be deduced from the graphs of cos x and sin x. He should also draw the graph of cot x from $x = -270°$ to $x = 270°$ and compare it with the graph of tan x.

Fig. 4 shows the graph of sec x, which, if moved 90° to the right, would become the graph of cosec x. Since

$|\sec x| \geqq 1$ and $|\operatorname{cosec} x| \geqq 1$, no part of either graph lies between the parallels $y = \pm 1$.

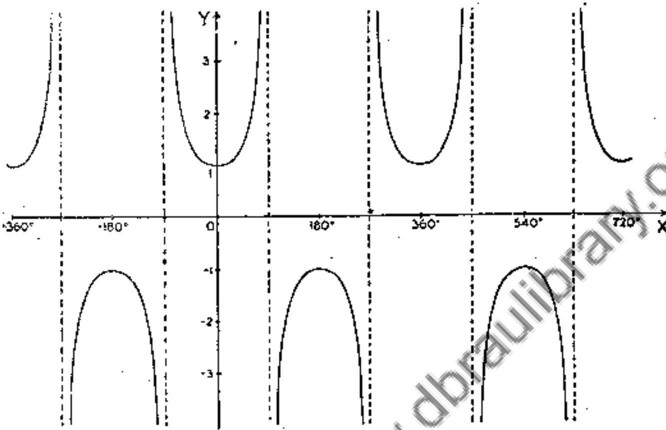


FIG. 4. $y = \sec x$.

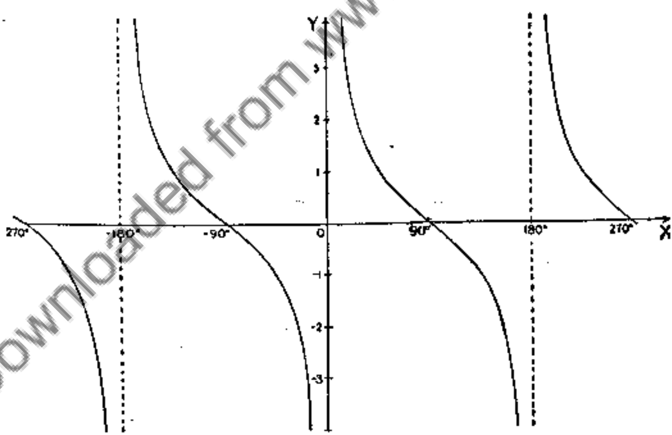


FIG. 5. $y = \cot x$.

Fig. 5 shows the graph of $\cot x$. Throughout its course the graph falls from left to right, for $\cot x$ always decreases as x increases, except when x goes through a value for which cot x has no value. The graph of cot x has vertical asymptotes

$$x = 0, \pm 180°, \pm 360°, \ldots$$

Graphs of cos 2x and sin 2x.—The functions cos $2x$ and sin $2x$ have period $180°$, since, when x increases by $180°$, $2x$ increases by $360°$, and therefore the values of cos $2x$ and sin $2x$ are unchanged. It follows that the graphs of cos $2x$ and sin $2x$ consist of an unlimited number of waves identical with the part which lies between $x = 0$ and $x = 180°$. Fig. 6 shows the two waves of the graph of

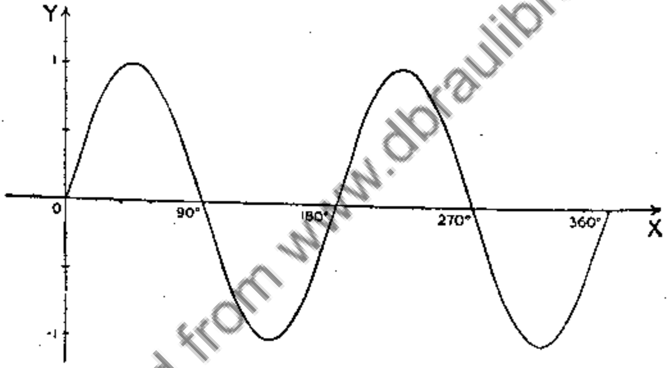


FIG. 6. $y = \sin 2x$.

sin $2x$ which lie in the range from $x = 0$ to $x = 360°$. The student should sketch the graph of cos $2x$ for the same range.

Similarly, the cosine and the sine of $3x$, $4x$, . . . have periods $360°/3$, $360°/4$, . . ., and their graphs for the range from $x = 0$ to $x = 360°$ consist of 3, 4, . . . identical waves.

Periods of tan nx, cos nx and sin nx.—Here, for convenience, the angles are expressed in radians. If x increases by π/n, nx increases by π and the value of tan nx is not changed. Hence tan nx has the period π/n.

If x increases by $2\pi/n$, nx increases by 2π, and the values of cos nx and sin nx are not changed. Hence cos nx and sin nx have the period $2\pi/n$.

When a function consists of several terms, each of which is periodic, any common multiple of the periods of the terms is a period of the function. The *least* common multiple is *the* period of the function.

Example.—Find the periods of the following functions:

(i) $2\cos 3x + 5\cos \frac{2}{3}x$;
(ii) $4\sin 3x - 3\cos 6x$;
(iii) $a\sin 3x + b\sin 5x$;
(iv) $\tan 2x - 4\tan 3x$.

Ans. (i) 6π; (ii) $2\pi/3$; (iii) 2π; (iv) π.

A function of the form

$$a_0 + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \ldots$$
$$+ b_1\sin x + b_2\sin 2x + b_3\sin 3x + \ldots$$

has period 2π, for the variable terms have periods 2π, $2\pi/2$, $2\pi/3$, . . ., of which the least common multiple is 2π.

§ 2. The Inverse Circular Functions

An equation of the form $y = f(x)$, which defines y explicitly as a function of x, may be regarded as defining x implicitly as a function of y. Two functions defined in this way are said to be *inverse* to each other. The usual notation for the function inverse to a given function f, is f^{-1}; thus, if $y = f(x)$, $x = f^{-1}(y)$.

For example, the equation $y = 2x + 3$ is equivalent to $x = \frac{1}{2}(y - 3)$. The two functions $2x + 3$ and $\frac{1}{2}(y - 3)$ are therefore inverse to each other.

It may happen that the equation $y = f(x)$ leads to more than one possible value of x in terms of y. The inverse function is then said to be *many-valued*. One of its values may be chosen as *principal value*, it being then understood that, unless the contrary is stated or implied, the symbol $f^{-1}(y)$ represents the principal value.

For example, if $y = x^2 \equiv f(x)$, then *either* $x = \sqrt{y}$ *or* $x = -\sqrt{y}$. The function inverse to x^2 is two-valued. The value $\sqrt{y}$ is chosen as principal value, and is represented by $f^{-1}(y)$.

Definitions.—The inverse circular functions are defined as follows:

if $y = \cos x$, then $x = \cos^{-1} y$;
if $y = \sin x$, then $x = \sin^{-1} y$;
if $y = \tan x$, then $x = \tan^{-1} y$;

and similarly for the other circular functions.

The symbol $\cos^{-1} y$ thus represents an angle whose cosine has the value y; $\sin^{-1} y$, an angle whose sine has the value y; $\tan^{-1} y$, an angle whose tangent has the value y; and so on. These new symbols should be read "cos minus one y," "sine minus one y," "tan minus one y," etc. Alternative symbols, arc cos y, arc sin y, arc tan y, etc., are sometimes used.

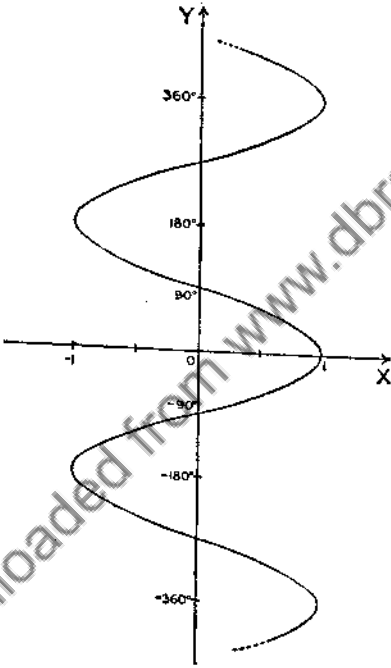


FIG. 7. $y = \cos^{-1} x$.

It should be noted that $\cos^{-1} y$ and $\sin^{-1} y$ are defined only for

$$-1 \leqq y \leqq 1,$$

since neither $\cos x$ nor $\sin x$ can be numerically greater than unity. Similarly, $\sec^{-1} y$ and $\operatorname{cosec}^{-1} y$ are defined only for $y \leqq -1$ and for $y \geqq 1$.

Graphs.—The graph of $\cos^{-1} x$ (Fig. 7) may be obtained from the graph of $\cos x$ (Fig. 1) merely by interchanging the labels of the co-ordinate axes, writing Y′OY for X′OX and X′OX for Y′OY; for the equation of the graph of

$\cos x$ is $y = \cos x$, or $x = \cos^{-1} y$, and this equation becomes $y = \cos^{-1} x$ when x and y are interchanged. After the interchange, however, the axes are not in the normal relative position, since the y-axis now makes a negative right angle with the x-axis. They may be brought into the normal position by rotating the whole diagram about the bisector of the angle XOY.

By a similar process the graphs of $\sin^{-1} x$ (Fig. 8) and $\tan^{-1} x$ (Fig. 9) may be derived from the graphs of $\sin x$ and $\tan x$.

The student should sketch the graphs of $\sec^{-1} x$, $\operatorname{cosec}^{-1} x$ and $\cot^{-1} x$.

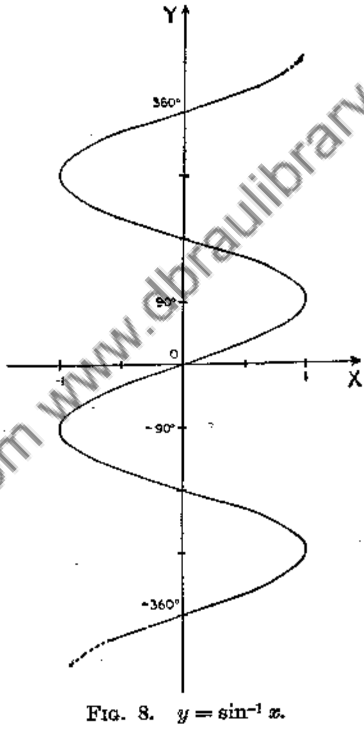


FIG. 8. $y = \sin^{-1} x$.

Principal Values.—Each of the inverse circular functions is many-valued. For example, if $\cos \alpha = x_1$, every angle of the set $\pm\alpha + n \cdot 360°$, where $n = 0, \pm 1, \pm 2, \ldots$, is an angle whose cosine is x_1, that is, a value of $\cos^{-1} x_1$. The graph of $\cos^{-1} x$ (Fig. 7) shows that, corresponding to any value of x in the range $-1 \leqq x \leqq 1$, there is an unlimited number of values of y or $\cos^{-1} x$, and that, in every case, one and only one of these values lies in the range $0 \leqq y \leqq 180°$. That value is chosen as the *principal value* of $\cos^{-1} x$.

Similarly, the graph of $\sin^{-1} x$ (Fig. 8) shows that, of the unlimited number of values of $\sin^{-1} x$, where

$-1 \leqq x \leqq 1$, one and only one lies in the range $-90° \leqq y \leqq 90°$. That value is chosen as the *principal value* of $\sin^{-1} x$.

Also, from Fig. 9, it is seen that, of the unlimited number of values of $\tan^{-1} x$, where x has any real value, one and only one lies in the range $-90° < y < 90°$. That value is chosen as the *principal value* of $\tan^{-1} x$.

The principal value of $\sec^{-1} x$ is the angle in the range

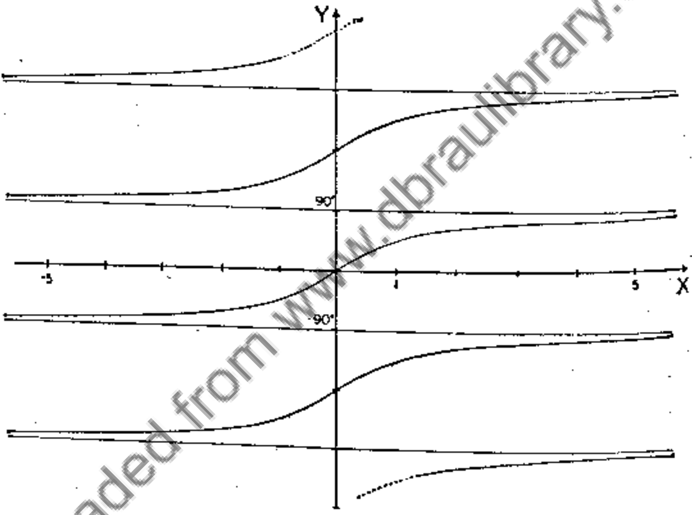


FIG. 9. $y = \tan^{-1} x$.

from 0 to 180°; and the principal values of $\operatorname{cosec}^{-1} x$ and $\cot^{-1} x$ are the angles in the range from $-90°$ to $90°$.

To sum up, the principal value of any inverse circular function is that value which is numerically least, the positive angle being taken when two, of opposite sign and equal size, are the smallest numerically.

The principal value should be understood in every case where it is not indicated or clearly implied that the general value is meant.

Example 1.—Verify the following statements:
$\cos^{-1}(1/2)=60^\circ$; $\sin^{-1}(1/2)=30^\circ$; $\tan^{-1}(1)=45^\circ$;
$\cos^{-1}(-1/2)=120^\circ$; $\sin^{-1}(-1/2)=-30^\circ$; $\tan^{-1}(-1)=-45^\circ$;
$\sec^{-1}(2/\sqrt{3})=30^\circ$; $\operatorname{cosec}^{-1}(2/\sqrt{3})=60^\circ$; $\cot^{-1}(\sqrt{3})=30^\circ$;
$\sec^{-1}(-\sqrt{2})=135^\circ$; $\operatorname{cosec}^{-1}(-\sqrt{2})=-45^\circ$;
$\cot^{-1}(-\sqrt{3})=-30^\circ$.

Example 2.—Using tables, find approximately the values of
(i) $\cos^{-1}(0{\cdot}62588)$; (ii) $\cos^{-1}(-0{\cdot}62588)$; (iii) $\sin^{-1}(0{\cdot}42813)$;
(iv) $\sin^{-1}(-0{\cdot}42813)$; (v) $\tan^{-1}(1{\cdot}35723)$; (vi) $\tan^{-1}(-1{\cdot}35723)$.
Ans. (i) $51^\circ\,15'$; (ii) $128^\circ\,45'$; (iii) $25^\circ\,21'$;
(iv) $-25^\circ\,21'$; (v) $53^\circ\,37'$; (vi) $-53^\circ\,37'$.

Example 3. Prove that
(i) $\cos^{-1}(-x)=\pi-\cos^{-1}x$; (ii) $\sin^{-1}(-x)=-\sin^{-1}x$;
(iii) $\tan^{-1}(-x)=-\tan^{-1}x$.

(i) Suppose that x is positive, and let $\theta=\cos^{-1}x$. Then θ is in the range from 0 to $\frac{1}{2}\pi$, and $x=\cos\theta$. Hence,

$$-x=-\cos\theta=\cos(\pi-\theta);$$

and therefore $\cos^{-1}(-x)=\pi-\theta$, this being the principal value since $\pi-\theta$ is in the range from $\frac{1}{2}\pi$ to π. Therefore,

$$\cos^{-1}(-x)=\pi-\cos^{-1}x.$$

If x is negative, let $x=-y$, so that y is positive. Then $\cos^{-1}(-y)=\pi-\cos^{-1}(y)$, by the result just proved. Hence, $\cos^{-1}(x)=\pi-\cos^{-1}(-x)$, which gives the same form as before.

If $x=0$, $\cos^{-1}(x)=\cos^{-1}(-x)=\frac{1}{2}\pi$, so that (i) is true for all values of x.

The identities (ii) and (iii) may be established in a similar manner.

Example 4. Prove that

$$\text{(i)}\quad \cos^{-1}x+\sin^{-1}x=\tfrac{1}{2}\pi;$$

$$\text{(ii)}\quad \tan^{-1}x+\cot^{-1}x=\begin{cases}\frac{1}{2}\pi, \text{ if } x>0,\\ -\frac{1}{2}\pi, \text{ if } x<0.\end{cases}$$

(i) Let x be positive, and let $\theta=\cos^{-1}x$. Then θ is in the range from 0 to $\frac{1}{2}\pi$, and $x=\cos\theta$. Hence $x=\sin(\frac{1}{2}\pi-\theta)$, and $\frac{1}{2}\pi-\theta$ is in the range from 0 to $\frac{1}{2}\pi$.

Therefore, $\sin^{-1}x=\frac{1}{2}\pi-\theta=\frac{1}{2}\pi-\cos^{-1}x$.

If x is negative, $-x$ is positive, and the result just proved gives

$$\cos^{-1}(-x)+\sin^{-1}(-x)=\tfrac{1}{2}\pi,$$

or

$$\pi-\cos^{-1}(x)-\sin^{-1}(x)=\tfrac{1}{2}\pi,$$

applying the results of *Example* 3 above.

If $x = 0$, $\cos^{-1} x = \frac{1}{2}\pi$, and $\sin^{-1} x = 0$.
Hence (i) is true for all values of x.
The identities (ii) may be established similarly.

Example 5.—Show that $\sin^{-1}(\frac{4}{5}) = \tan^{-1}(\frac{4}{3}) = \cos^{-1}(\frac{3}{5})$, and that $\sin^{-1}(-\frac{4}{5}) = \tan^{-1}(-\frac{4}{3}) = \cos^{-1}(-\frac{3}{5}) - \pi$.

§ 3. Graphical Solution of Equations

Roots of equations of the form $f(x) = \phi(x)$, where $f(x)$ and $\phi(x)$ are any functions of x, may be determined approximately by drawing in the same diagram, with the same axes and scales, the graphs of $f(x)$ and $\phi(x)$. The abscissæ of points common to the two graphs are roots of the given equation; for, if (x_1, y_1) is any common point, its co-ordinates satisfy the equations $y = f(x)$ and $y = \phi(x)$. Hence $y_1 = f(x_1)$ and $y_1 = \phi(x_1)$.

Therefore $f(x_1) = \phi(x_1)$; that is, x_1 is a root of the equation $f(x) = \phi(x)$.

As a simple special case, the abscissæ of the points where the graph of $f(x)$ meets the x-axis are roots of the equation $f(x) = 0$.

Example 1.—Find approximately by a graphical method the roots of the equation $\cos 2x = 2 \sin x$ which are in the range $0 \leqq x \leqq 180°$.

TABLE 4

x	0 180°	10° 170°	20° 160°	30° 150°	40° 140°	50° 130°	60° 120°	70° 110°	80° 100°	90°
$\cos 2x$	1	0·94	0·77	0·5	0·17	−0·17	−0·5	−0·77	−0·94	1
$2 \sin x$	0	0·35	0·68	1	1·29	1·53	1·73	1·88	1·97	2

Table 4 shows the values of $\cos 2x$ and $2 \sin x$, correct to two places of decimals, for values of x from 0 to 180° by steps of 10°. The graphs of $y = \cos 2x$ and $y = 2 \sin x$ are shown in Fig. 10. The abscissæ of the two points of intersection, 21·5° and 158·5°, are the required roots.

If the graphs are drawn accurately on squared paper with scales of (say) 1 inch = 20° on the x-axis and 1 inch = 0·5 unit on the y-axis, it should be possible to read off the roots to within one-fifth of a degree.

If greater accuracy is required, a much larger scale can be used. For example, in order to find the value of the first root more accurately, the two graphs might be drawn from $x = 10°$ to $x = 30°$, taking intervals of 1°, and using as scales 1 inch = 2° on the x-axis and 1 inch = 0·1 unit on the y-axis.

The student may verify, by sketching the two graphs from $x = 180°$ to $x = 360°$, that there are no points of intersection

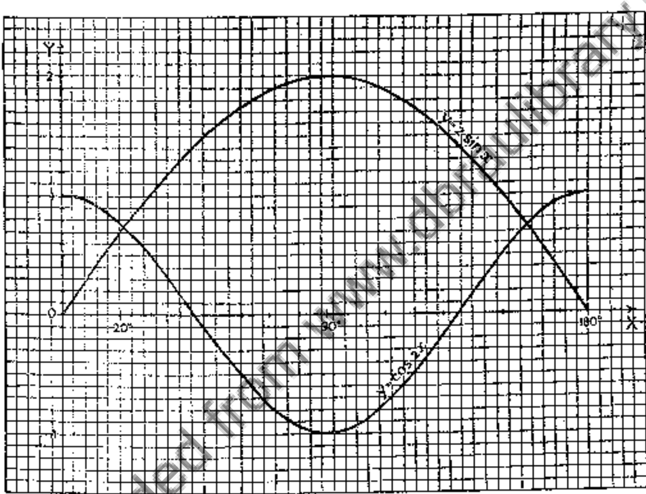


FIG. 10.

in that range, and that the values found are the only roots in the range from $x = 0$ to $x = 360°$.

Example 2.—Solve graphically the equation $x = \cos x$, where the angle is expressed in radians.

Since $-1 \leqq \cos x \leqq 1$, any root of this equation lies between -1 and 1. Also, if $-1 < x < 0$, $\cos x$ is positive, for the angle is between $-57° \ 17{\cdot}7'$ and zero; and therefore the equation cannot be satisfied. Hence all roots of the equation lie between 0 and 1, and it is sufficient to draw the graphs $y = x$ and $y = \cos x$ for the range $0 < x < 1$. For convenience the range $0 < x < \frac{1}{3}\pi$ is taken.

The values of $\cos x$, correct to three places of decimals, for values of x from 0 to $\frac{1}{3}\pi$ by steps of $\frac{1}{18}\pi$, are shown in Table 5. For the straight line graph $y = x$ only two points are necessary:

TABLE 5.

x . .	0	$\frac{1}{18}\pi$	$\frac{2}{18}\pi$	$\frac{3}{18}\pi$ 0·524	$\frac{4}{18}\pi$ 0·698	$\frac{5}{18}\pi$	$\frac{6}{18}\pi$ 1·047
Degrees	0	10	20	30	40	50	60
$\cos x$.	1	0·985	0·940	0·866	0·766	0·643	0·5

a third is plotted as a check. The values of x at the three points are shown in Table 5 correct to three places of decimals.

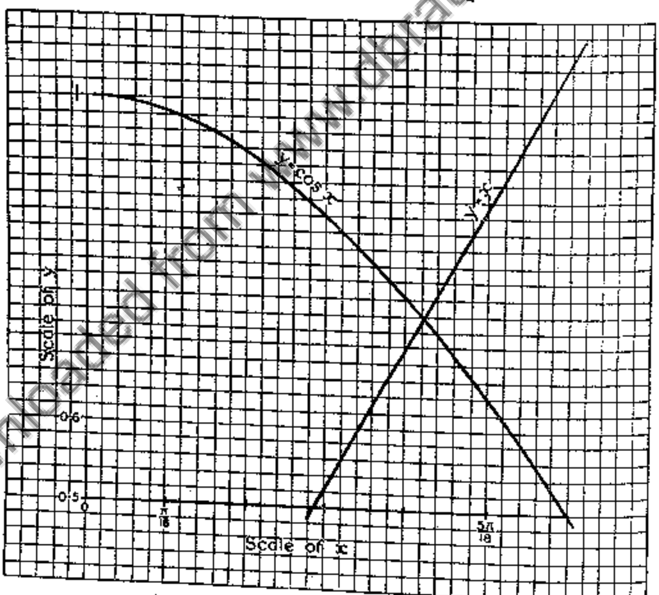


FIG. 11.

The graphs are shown in Fig. 11. It is seen that the equation $x = \cos x$ has only one real root, namely $4{\cdot}23 \times \frac{1}{18}\pi$, or approximately 0·738. The form $4{\cdot}23 \times \frac{1}{18}\pi$ is convenient if the angle is to be converted into sexagesimal measure, as is usually the case when an equation of this type arises out of a practical problem. The measure in degrees of the angle which satisfies the given equation is $4{\cdot}23 \times 10°$, or $42{\cdot}3°$.

§ 4. Area of a Segment of a Circle

Let the chord AB (Fig. 12) of a circle of radius r subtend an angle θ radians at the centre O. Then AB divides the circle into a minor segment ABPA, whose central angle is θ radians, and a major segment BAQB, whose central angle is $(2\pi - \theta)$ radians.

Let AD be the perpendicular from A to OB. Then $AD = r \sin AOD$, from the right-angled triangle ODA.

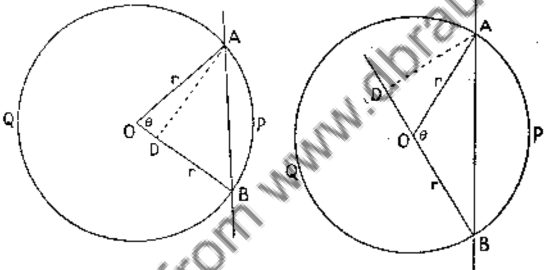


FIG. 12.

Now, if $\theta < \frac{1}{2}\pi$, D is on OB and $\angle AOD = \theta$; while, if $\theta > \frac{1}{2}\pi$, D is on BO produced and $\angle AOD = \pi - \theta$. In both cases, therefore, $\sin AOD = \sin\theta$, and so $AD = r\sin\theta$. Again, if $\theta = \frac{1}{2}\pi$, D coincides with O, and $AD = r = r\sin\theta$. Hence in all cases the area of the triangle AOB, namely $\frac{1}{2}OB \,.\, AD$ is $\frac{1}{2}r^2 \sin\theta$ square units.

Now the minor segment ABPA

$$= \text{the sector OBPAO} - \triangle AOB$$
$$= \tfrac{1}{2}r^2\,\theta - \tfrac{1}{2}r^2 \sin\theta\,;$$

and the major segment BAQB

$$= \text{the sector OAQBO} + \triangle AOB$$
$$= \tfrac{1}{2}r^2\,(2\pi - \theta) + \tfrac{1}{2}r^2 \sin\theta$$
$$= \tfrac{1}{2}r^2\,\phi - \tfrac{1}{2}r^2 \sin\phi,$$

where $\phi = 2\pi - \theta.$

Thus in all cases, whether the segment be minor or major, the area is given by the expression $\frac{1}{2}r^2(\theta - \sin\theta)$, where r is the radius of the circle, and θ is the number of radians in the central angle of the segment.

Example.—Find approximately the size of the angle subtended at the centre of a circle by a chord which divides the circle into segments whose areas are in the ratio 2 : 1.

Let x be the number of radians in the required angle. Then, if r is the radius of the circle,

$$\tfrac{1}{2}r^2(x - \sin x) = \tfrac{1}{3}\pi r^2.$$

Hence,

$$\sin x = x - \tfrac{2}{3}\pi \quad . \quad . \quad . \quad . \quad (1)$$

A rough sketch of the graphs $y = \sin x$ and $y = x - \frac{2}{3}\pi$ shows that the root of (1) is between $\frac{2}{3}\pi$ and π. These graphs can then be drawn to as large a scale as is convenient, and the root can be read off. A suitable table of values is given below, the drawing of the graphs being left as an exercise to the reader.

x	$12\pi/18$	$13\pi/18$	$14\pi/18$	$15\pi/18$	$16\pi/18$	$17\pi/18$	$18\pi/18$
Degrees	120	130	140	150	160	170	180
$\sin x$.	0·866	0·766	0·643	0·5	0·342	0·174	0
$x - \frac{2}{3}\pi$	0	—	—	0·524	—	—	1·047

The solution of (1) is $x \fallingdotseq 14{\cdot}93 \times \pi/18$. The size of the angle in degrees is therefore approximately 149·3°.

EXAMPLES IV

1. Draw the graphs of the following functions for the range from $x = 0$ to $x = 360°$.

(i) $\cos 2x$; (ii) $\cos 3x$; (iii) $\cos 4x$;
(iv) $\sin 3x$; (v) $\sin 4x$; (vi) $\tan 2x$.

2. Draw in the same diagram the graphs of $\cos 2x$ and $\sin(x - 35°)$ for the range from $x = 0$ to $x = 180°$, and hence find the roots of the equation $\cos 2x = \sin(x - 35°)$ which lie within that range.

Ans. $x = 41{\cdot}7°$ or $161{\cdot}7°$.

3. By means of the graphs of $\tan x$ and $6 - 5 \sin x$, find the angle in the range $0 < x < 80°$, which satisfies the equation $\tan x + 5 \sin x = 6$.

Ans. $59 \cdot 4°$.

4. Show that the equation $\sin 2x = 1 - \cos x$ cannot be satisfied by any value of x between $90°$ and $360°$.

Tabulate, correct to three places of decimals, the values of $\sin 2x$ and $1 - \cos x$ for $x = 0, 10°, 20°, \ldots, 90°$; and, by drawing the graphs of these functions, find the values of x in the range $0 \leq x \leq 360°$ which satisfy the given equation.

Ans. $0, 69 \cdot 6°, 360°$.

5. Draw in one diagram the graphs of $\cos^2 x$ and $(\sin \frac{1}{2}x - 0 \cdot 4)$ for the range from $x = 0$ to $x = 180°$, and hence find the values of x in that range which satisfy the equation

$$\cos^2 x - \sin \tfrac{1}{2}x + 0 \cdot 4 = 0.$$

Deduce the solutions in the range from $x = 180°$ to $x = 360°$.

Ans. $67 \cdot 0°, 136 \cdot 7°, 223 \cdot 3°, 293 \cdot 0°$.

6. Draw a rough graph of $y = 4 \sin x - 3 \cos x$, and show that there is only one real root of the equation

$$3x + 4 \sin x = 3 \cos x,$$

where x is the number of radians of angle. Find that root by means of an accurate drawing of part of the graph.

Ans. $x = 0 \cdot 401 = 0 \cdot 128\pi$.

7. Show graphically that the equation $\cos x = x - 1$, where x is in radians, has only one real root and determine that root correct to two places of decimals.

Ans. $x = 1 \cdot 28$.

8. Solve graphically the equation $3 \cos x = x - \frac{1}{2}\pi$, where x is in radians, and convert the solution into sexagesimal measure.

Ans. $x = 1 \cdot 44$; $82 \cdot 5°$.

9. A chord of a circle is 24 inches long and is 5 inches distant from the centre of the circle; find the area of the smaller segment which it cuts off.

Ans. 138·8 square inches.

10. Calculate the area of a circular segment which contains an angle of 125° and stands upon a chord 12 inches long.

Ans. 26·29 square inches.

11. The chord AC of a circular segment ABC is 12 inches long, and the angle ABC in the segment is 35° 24′; find the length of the arc ABC to the nearest tenth of an inch, and the area of the segment to the nearest square inch.

Ans. 52·3 inches; 321 square inches.

12. Two circles of radii 3 inches and 4 inches are described with their centres 5 inches apart. Calculate the area common to the two circles.

Ans. 6·641 square inches.

13. Two equal circles of radius 10 inches have their centres 14 inches apart. Calculate the overlapping area.

Ans. 59·11 square inches.

14. Calculate the area common to two equal circles of radius 3 feet whose centres are 4 feet apart.

Ans. 6·195 square feet.

15. The common chord of two intersecting circles is a diameter of one of them, and is equal in length to the radius r of the other. Show that the area common to the two circles is $\frac{1}{24}r^2(7\pi - 6\sqrt{3})$.

16. Circles with centres A and B, and radii a and b, respectively, cut orthogonally at P. Show that, if α denotes $\angle$PAB, the area common to the circles is

$$(a^2 - b^2)\alpha + \tfrac{1}{2}\pi b^2 - ab.$$

17. The section of a subway tunnel is a major segment of a circle. If the base chord is 6 feet and the greatest height is 10 feet, calculate, in cubic yards, the volume of the tunnel per 100 yards of length.

Ans. 996 cubic yards.

18. Find the volume of water contained in a horizontal cylindrical boiler, 6 feet in diameter and 20 feet long, when the greatest depth of the water is 4 feet.

Ans. 400·5 cubic feet.

19. Through a point on the circumference of a circle of radius 5 inches, two equal chords are drawn, inclined to each other at an angle of 50°. Find the areas of the three parts into which the circle is thus divided.

Ans. 18·79, 40·97 and 18·79 square inches.

20. In a circle of radius 17 inches a chord is placed at a perpendicular distance of 8 inches from the centre, and on this chord as diameter a circle is described. Calculate the area of that part of the larger circle which is outside the smaller.

Ans. 362·1 square inches.

21. Two radii OA, OB of a circle contain an acute angle. A semicircle on OA as diameter cuts OB in C. Show that the arc CA of this semicircle bisects the area bounded by CB, CA and the arc AB.

22. On OA, a radius of a circle whose centre is O, an equilateral triangle OAB is described. Show that the area of that part of the circumcircle of the triangle OAB which is external to the given circle is $\frac{1}{18}r^2(3\sqrt{3} - \pi)$, where r is the radius of the given circle.

23. An arc ABC of a circle of radius r subtends an angle of 75° at the centre. On AC a semicircle ADC is described outside the circle ABC. Calculate the area ABCDA.

Ans. $0{\cdot}41r^2$.

24. AB = AC = 12·5 inches, and ∠BAC = 35° 30′. A circle is described to touch AB at B and AC at C. Calculate the area of that segment of the circle which lies outside the triangle ABC.

Ans. 34·76 square inches.

25. Points A, B, C are taken in order on a straight line, such that AB = 5 inches and BC = 3 inches. On AC and BC as diameters semicircles are drawn. The tangent from A to the smaller semicircle meets the larger in D. Calculate the area of the segment cut off by the chord AD.

Ans. 17·81 square inches.

26. C is the point of trisection nearer A of a straight line AB of length 6 inches, and semicircles are drawn on AC and CB as diameters. The tangent AD from A to the larger semicircle cuts the smaller in E. Calculate (i) the area of the segment cut off by AE, and (ii) the area enclosed between D and the circles.

Ans. (i) $\frac{1}{3}\pi - \frac{1}{4}\sqrt{3} \fallingdotseq 0{\cdot}614$ square inches;
(ii) $\frac{7}{4}\sqrt{3} - \frac{5}{6}\pi \fallingdotseq 0{\cdot}413$ square inches.

27. The hydraulic mean depth is defined as the ratio of the cross-sectional area of water to the wetted perimeter. If r is the radius of the circular section of a cylindrical pipe, and θ is the radian measure of the angle subtended at the centre by the wetted perimeter, find the hydraulic mean depth.

Ans. $\frac{1}{2}r(1 - \sin\theta/\theta)$.

28. A cylindrical boiler l feet long, and of r feet radius, with its axis horizontal, contains water to a depth of d feet ($d > r$), the remainder of the space being occupied by steam. Prove that the volume of the steam space is

$$l\left\{r^2\cos^{-1}\left(\frac{d-r}{r}\right) - (d-r)\sqrt{(2rd - d^2)}\right\} \text{ cubic feet.}$$

29. AC is a diameter of a circle whose centre is O. A chord AB is drawn in such a way that the area of the smaller segment which it cuts off is equal to the area of the triangle BOC. If

θ is the radian measure of $\angle$AOB, show that $\sin\theta = \frac{1}{2}\theta$. Solve this equation graphically, and show that $\angle$AOB $= 108{\cdot}6°$ approximately.

30. From one end A of the diameter AB of a semicircle a chord AP is drawn to divide the area of the semicircle into two equal parts. Show that, if $\angle$BAP $= \theta$ radians,

$$\cos\left(\tfrac{1}{2}\pi - 2\theta\right) = \tfrac{1}{2}\pi - 2\theta.$$

Solve the equation $\cos x = x$ graphically, and hence find approximately the size of $\angle$BAP in degrees.

Ans. $23{\cdot}8°$.

31. The area of the minor segment cut off by a chord AB of a circle whose centre is O, is three times the area of the triangle AOB. If the number of radians in $\angle$AOB is x, show that $\sin x = \frac{1}{4}x$, and determine by a graphical method the size of $\angle$AOB in degrees.

Ans. $141{\cdot}8°$.

32. BC is a diameter of a circle whose centre is O. A point A is taken on the circle such that the areas of the minor segments cut off by AB and AC are in the ratio 3 : 1. If x is the radian measure of $\angle$COA, show that $\sin x = 2x - \frac{1}{2}\pi$. Determine $\angle$COA in degrees to the nearest tenth of a degree.

Ans. $72{\cdot}3°$.

33. In a circle whose centre is O, the chord AB and the diameter CD are parallel and in the same sense, and the smaller segment cut off by AB is equal in area to the sector BOD. Find approximately the size of $\angle$AOB.

Ans. $97{\cdot}8°$.

34. OA and OB are perpendicular radii of a circle, and C is a point on the shorter arc AB such that the area of the sector BOC is twice that of the segment cut off by AC. Determine the size of $\angle$BOC approximately.

Ans. $25{\cdot}5°$.

CHAPTER V

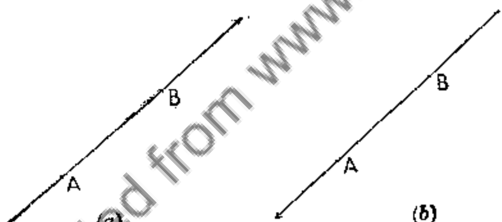
ORTHOGONAL PROJECTION

§ 1. Directed Lines

One direction on a straight line may be chosen arbitrarily as the positive direction; this direction will be indicated by an arrow-head. The opposite direction is then the negative direction. Lengths measured in the positive and negative directions are positive and negative respectively. In Fig. 1(a), AB is positive, BA negative, while in Fig. 1(b), AB is negative, BA positive. In all cases

$$BA = -AB.$$

The co-ordinate axes in analytical geometry are examples of directed lines.



(a) (b)

Fig. 1.

Theorem I.—If A, B, C are any three points on a directed line

$$AB = CB - CA.$$

Case I.—Let AB be positive. There are then three possible arrangements for A, B, and C, as indicated in Fig. 2(a), (b), and (c).

In (a) $\quad AB = CB - CA;$

in (b) $\quad AB = AC + CB = CB - CA;$

and in (c) $\quad AB = AC - BC = CB - CA.$

Thus the theorem is always true if AB is positive.

CASE II.—Let AB be negative; then BA is positive. Hence, by Case I,

$$BA = CA - CB$$

and therefore

$$AB = -BA = CB - CA,$$

so that the formula holds in all cases.

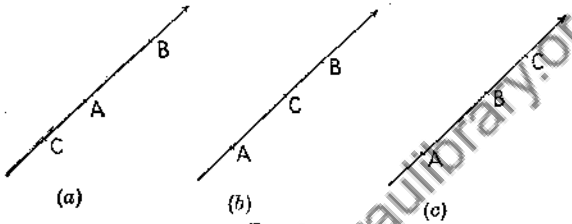


FIG. 2.

THEOREM II.—If A, B, C, . . ., L, M, N are any points (arranged in any order) on a directed line

$$AB + BC + CD + \ldots + LM + MN = AN.$$

For, if O is any point on the line,

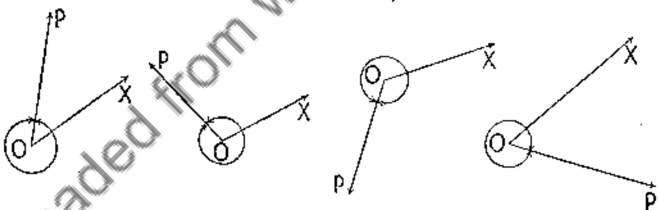


FIG. 3.

$$\begin{aligned} AB + BC + CD + \ldots + LM + MN &= OB + OC + OD + \ldots + OM + ON \\ &\quad - OA - OB - OC - \ldots - OL - OM \\ &= ON - OA = AN. \end{aligned}$$

COROLLARY.—$AB + BC + \ldots + LM + MN + NA = 0.$

Angle between Two Directed Lines.—If OX and OP are two directed lines the angle that OP makes with OX, that is, the angle traced out by a radius vector which revolves about O from the initial direction OX to the final direction

OP, is called the angle XOP. From Fig. 3 it may be seen that there is always one positive angle XOP less than four right angles, and one negative angle XOP numerically less than four right angles, and that these two angles differ by four right angles. All the other angles XOP may be obtained from either of these by adding or subtracting multiples of four right angles. The equation

$$\angle POX = -\angle XOP$$

is always true, provided that the angles referred to are those which are numerically equal.

§ 2. Orthogonal Projection

If A′ is the foot of the perpendicular from a point A on a directed line X′X, A′ is said to be the orthogonal projection of A on X′X. If A′ and B′ are the (orthogonal) projections * of two points, A and B, of a directed line, on X′X, A′B′ is the projection of the segment AB on X′X. The line X′X is called the *axis of projection*.

For instance, if P is the point (x, y) in the co-ordinate plane, and O the origin, the projections of OP on X′X and Y′Y are x and y, respectively.

From the definitions of a cosine and a sine, these projections may be expressed as $r \cos \theta$ and $r \sin \theta$, respectively, where $\angle XOP = \theta$ and r is the length of OP.

Note.—The abbreviation "Proj." will frequently be employed for "Projection."

THEOREM I.—

$$\text{Proj. of BA on X'X} = -\text{Proj. of AB on X'X.}$$

For † $\quad$ Proj. of BA $= B'A'$,

$\quad\quad$ Proj. of AB $= A'B'$,

and $\quad\quad B'A' = -A'B'$.

* In what follows the word "orthogonal" will be omitted, as this is the only type of projection with which we are concerned. It will be further assumed that all the points dealt with are coplanar.

† Where there is no dubiety regarding the axis of projection it is not necessary to refer to it.

THEOREM II.—If A, B, C, . . ., L, M, N are any points whatever, the sum of the projections of AB, BC, . . ., LM, MN on X′X is equal to the projection of AN on X′X.

For, if the projections of A, B, C, . . ., N are A′, B′, C′, . . ., N′, the sum of the projections of AB, BC, . . ., MN is

$$A'B' + B'C' + \ldots + M'N' = A'N',$$

by § 1, Theorem II, and this is the projection of AN.

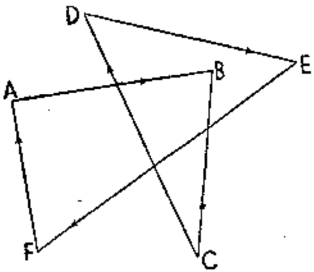


FIG. 4.

COROLLARY.—The sum of the projections on any directed line of the sides, taken in order, of any closed polygon, is zero.

Note.—The sides of the polygon may intersect, as in Fig. 4.

THEOREM III.—The projections on any directed line of two straight lines which are equal, parallel and measured in the same direction, are equal.

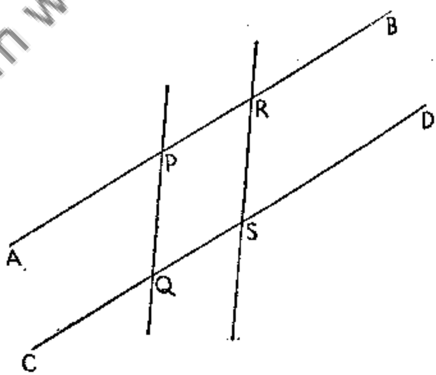


FIG. 5.

Note.—The *directions* of parallel lines can be compared as follows. Let AB and CD (Fig. 5) be two such lines, and let a straight line cut them in P and Q respectively. Let a second straight line drawn parallel to PQ cut them in R and S respectively. Then the segments PR and QS are said to be measured in the same direction. Two straight lines which are equal, parallel, and measured in the same direction are said to be *equivalent*. Thus, if ABCD (Fig. 6) is a parallelogram, AB and DC are equivalent, and AD and BC are also equivalent.

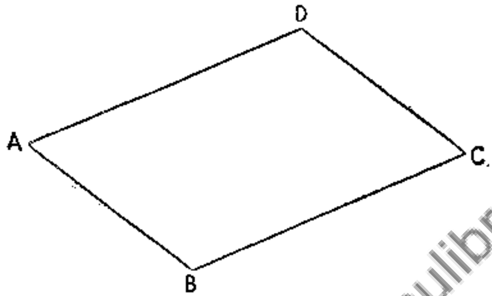


FIG. 6.

The theorem may be proved as follows. Let AB and CD (Fig. 7) be the two equivalent lines, X′X the axis of projection. Draw perpendiculars AA′, BB′, CC′, DD′ to X′X, and through A′ and C′ draw lines parallel to AB to

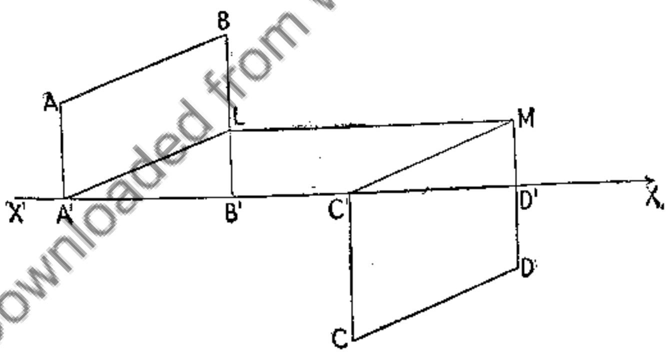


FIG. 7.

meet BB′ and DD′ (or these lines produced) in L and M respectively. Join LM.

Then A′L and C′M, being equivalent to AB and CD, are equivalent to each other. Thus A′C′ML and consequently

B′D′ML are parallelograms, so that A′C′ and B′D′, which are both equivalent to LM, are equivalent to each other. Hence

$$A'B' + B'C' = A'C' = B'D' = B'C' + C'D',$$

and therefore $$A'B' = C'D',$$

from which the theorem follows.

Note.—These theorems still hold if the points and lines are not coplanar. In the proofs the perpendiculars to X′X should be replaced by planes perpendicular to X′X, and in Theorem III A′L and C′M meet the perpendicular planes through B and D in L and M.

THEOREM IV.—If AB is a segment of a directed line, and if the directed line makes an angle θ with the directed line X′OX, the projection of AB on X′OX is equal to AB cos θ.

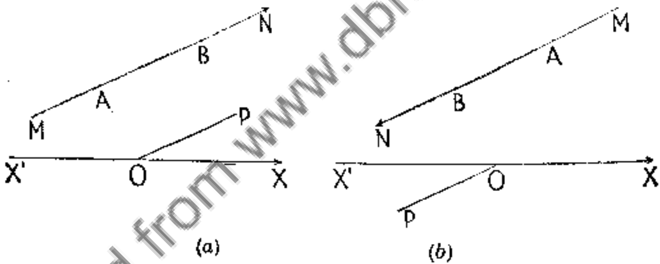


FIG. 8.

CASE I.—Let AB, as in Fig. 8(*a*), (*b*), be a positive segment of the directed line MN. Draw OP parallel to, and in the same direction as, MN and equal to AB. Then, since $\angle XOP = \theta$,

$$\text{Proj. of AB on X'X} = \text{Proj. of OP on X'X}$$
$$= \text{OP} \cos \theta,$$

from the definition of a cosine, OP being positive,

$$= \text{AB} \cos \theta.$$

Note.—Since $\cos(\theta + n \,.\, 360°) = \cos \theta$, where n is any integer, the theorem is true for any angle XOP.

CASE II.—Let AB be negative: then

$$\begin{aligned}\text{Proj. of AB on X'X} &= -\text{ Proj. of BA on X'X}\\ &= -(\text{BA}\cos\theta),\text{ by Case I.}\\ &= \text{AB}\cos\theta.\end{aligned}$$

COROLLARY.—If AB is a segment of MN, a directed line which makes an angle θ with the directed line X'OX, and if Y'OY is a directed line drawn so that angle XOY is a positive right angle, the projection of AB on Y'OY is equal to AB $\sin\theta$.

The angle made by MN with Y'OY is $-90^\circ + \theta$: for a radius-vector with OY as initial position after revolving

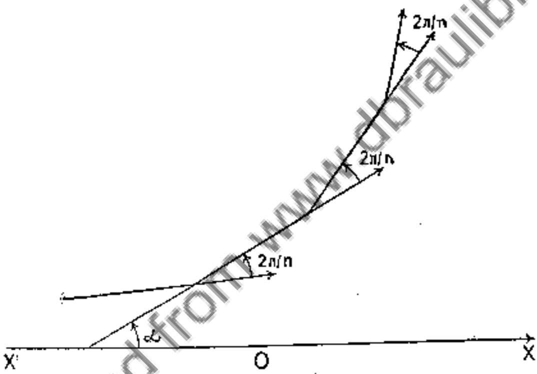


FIG. 9.

through an angle -90° attains the direction OX; on revolving through a further angle θ it is parallel to and in the same direction as MN.

Hence, by Theorem IV,

$$\text{Proj. of AB on Y'OY} = \text{AB}\cos(-90^\circ + \theta) = \text{AB}\sin\theta.$$

Alternatively, this result may be established in the same way as Theorem IV, it being noted that, by the definition of the sine, the projection of OP on Y'Y is OP $\sin\theta$.

Example 1.—Consider a regular n-sided polygon (Fig. 9), with sides of length l, and let the positive directions of the sides correspond to the positive (counter-clockwise) direction of tracing the polygon. Let one side make an angle α with X′OX: the other sides, taken in order, make angles $\alpha + 2\pi/n$, $\alpha + 4\pi/n, \ldots, \alpha + 2(n-1)\pi/n$ with X′OX. Then the sum of their projections on X′OX is

$$l\cos\alpha + l\cos\left(\alpha + \frac{2\pi}{n}\right) + l\cos\left(\alpha + \frac{4\pi}{n}\right) + \ldots + l\cos\left\{\alpha + \frac{(2n-2)\pi}{n}\right\},$$

so that, by Theorem II., corollary,

$$\cos\alpha + \cos\left(\alpha + \frac{2\pi}{n}\right) + \cos\left(\alpha + \frac{4\pi}{n}\right) + \ldots + \cos\left\{\alpha + \frac{(2n-2)\pi}{n}\right\} = 0.$$

Similarly, by projecting on Y′OY, it can be shown that

$$\sin\alpha + \sin\left(\alpha + \frac{2\pi}{n}\right) + \sin\left(\alpha + \frac{4\pi}{n}\right) + \ldots + \sin\left\{\alpha + \frac{(2n-2)\pi}{n}\right\} = 0.$$

Example 2.—Take one side of a regular seven-sided polygon as x-axis, and the mid-point of the side as origin, and prove, by projecting three sides on the x and y axes, that,

$$\text{(i)}\ \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\tfrac{1}{2},$$

$$\text{(ii)}\ \sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{6\pi}{7} = \tfrac{1}{2}\cot\frac{\pi}{14}.$$

Example 3.—OPQRS is a regular pentagon, O being the origin, and $\angle XOP = A$. Prove by projection that

$$\cos A + \cos(A + 72°) + \cos(A - 72°) = \cos(A + 36°) + \cos(A - 36°).$$

Example 4.—P, A, B, C, . . . , M, N are $n + 1$ points taken in order on the circumference of a circle of centre O and radius r, such that the arcs PA, AB, BC, . . . , MN all subtend equal angles β at O. If $\beta < 2\pi/n$, show that $PA = 2r\sin\frac{1}{2}\beta$ and that $PN = 2r\sin\frac{1}{2}n\beta$.

If P is the origin, and $\angle XPA = \alpha$, prove, by projecting on the x and y axes, that

$$\text{(i)}\ \sin\tfrac{1}{2}\beta[\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha + 2\beta) + \ldots + \cos\{\alpha + (n-1)\beta\}] = \sin\left(\frac{n}{2}\beta\right)\cos\left(\alpha + \frac{n-1}{2}\beta\right),$$

(ii) $\sin \frac{1}{2}\beta[\sin \alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \ldots + \sin\{\alpha + (n-1)\beta\}]$

$$= \sin\left(\frac{n}{2}\beta\right) \sin\left(\alpha + \frac{n-1}{2}\beta\right).$$

§ 3. Traverses

A set of straight lines OA, AB, BC, . . ., MN in the (x, y) plane, taken in order, are said to form a *traverse*. The length and direction of ON can be determined by projecting OA, AB, . . ., MN on the x and y axes, and applying Theorems II and IV of the previous section. The directions OA, AB, . . ., MN are usually taken to be positive.

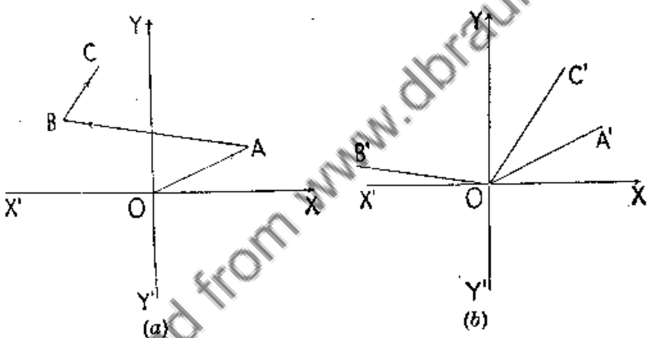


FIG. 10.

Example 1.—Consider the traverse OABC (Fig. 10(*a*)), where

$$\angle XOA = 25° \; 12', \quad \angle OAB = -33° \; 15', \quad \angle ABC = 64°,$$
$$OA = 13, \quad AB = 23, \quad BC = 8.$$

Draw OA′, OB′, OC′ (Fig. 10(*b*)) parallel to OA, AB, BC respectively, and determine the angles made by these lines with OX.

$$\angle XOA' = \angle XOA = 25° \; 12'.$$
$$\angle A'OB' = 180° - \angle BAO$$
$$= 180° - 33° \; 15' = 146° \; 45'.$$

Therefore $\angle XOB' = \angle XOA' + \angle A'OB' = 171° \; 57'$,

$\angle B'OC' = -(180° - \angle ABC) = -116°$;

so that $\angle XOC' = \angle XOB' + \angle B'OC' = 55° \; 57'$.

Let $OC = r$, $\angle XOC = \theta$; then, by § 2, Theorems II. and IV., the projection of OC on X′X is

$$\begin{aligned} r\cos\theta &= \text{sum of projections of OA, AB, and BC on X'X} \\ &= 13\cos(25^\circ\ 12') + 23\cos(171^\circ\ 57') + 8\cos(55^\circ\ 57') \\ &= 13 \times \cdot 90483 + 23 \times -\cdot 99014 + 8 \times \cdot 55992 \\ &= 11\cdot 7628 - 22\cdot 7732 + 4\cdot 4794 \\ &= -\,6\cdot 5310. \end{aligned}$$

Similarly, by projecting on Y′Y, it can be shown that

$$\begin{aligned} r\sin\theta &= 13\sin(25^\circ\ 12') + 23\sin(171^\circ\ 57') + 8\sin(55^\circ\ 57') \\ &= 13 \times \cdot 42578 + 23 \times \cdot 14003 + 8 \times \cdot 82855 \\ &= 5\cdot 5351 + 3\cdot 2207 + 6\cdot 6284 \\ &= 15\cdot 3842. \end{aligned}$$

Thus θ is the angle in the second quadrant for which

$$\tan\theta = -\frac{15\cdot 3842}{6\cdot 5310},$$

so that
$$\begin{aligned} \log\tan(180^\circ - \theta) &= \log 15\cdot 3842 - \log 6\cdot 5310 \\ &= 1\cdot 18709 - \cdot 81498 \\ &= \cdot 37211 = \log\tan 67^\circ\ 0', \end{aligned}$$
and therefore
$$\theta = 180^\circ - 67^\circ\ 0' = 113^\circ\ 0'.$$

Again,
$$\log r + \log\sin 113^\circ\ 0' = \log 15\cdot 3842,$$
so that
$$\begin{aligned} \log r &= 1\cdot 18709 - \bar{1}\cdot 96403 \\ &= 1\cdot 22306 = \log 16\cdot 71. \end{aligned}$$

Thus
$$r = 16\cdot 71.$$

Note.—The angles which AB and BC make with OX might be found as follows.

Since OA makes with OX an angle $25^\circ\ 12'$, AO makes with OX an angle $25^\circ\ 12' + 180^\circ = 205^\circ\ 12'$: but $\angle OAB = -33^\circ\ 15'$; therefore AB makes with OX an angle $205^\circ\ 12' - 33^\circ\ 15' = 171^\circ\ 57'$.

Hence BA makes with OX an angle $171^\circ\ 57' + 180^\circ = 351^\circ\ 57'$: but $\angle ABC = 64^\circ$; therefore BC makes with OX an angle $415^\circ\ 57'$ or $55^\circ\ 57'$.

Example 2.—A circle touches the x-axis at the origin O, and cuts the y-axis again at a point C above the origin; A and B are points on the circumference such that the order O, A, B, C is counter-clockwise. If $\angle AOC = \theta$, $\angle OCB = \phi$, find the angles which AB, BC make with OX, and show that

$$\begin{aligned} a\sin\theta + b\sin(\theta - \phi) &= c\sin\phi, \\ a\cos\theta + b\cos(\theta - \phi) + c\cos\phi &= d, \end{aligned}$$

where a, b, c, d are the lengths of OA, AB, BC, OC respectively.

Ans. $\frac{1}{2}\pi - \theta + \phi$, $\frac{1}{2}\pi + \phi$.

EXAMPLES V

1. Prove the identity

$$\sin A + \sin(A + \tfrac{2}{3}\pi) + \sin(A + \tfrac{4}{3}\pi) = 0$$

by projecting the sides of an equilateral triangle.

2. The side BC of a positive triangle ABC makes an angle θ with X′X. Prove that

$$a\cos\theta = b\cos(C - \theta) + c\cos(B + \theta).$$

Deduce that in any triangle

$$a\sin A - b\sin B = c\sin(A - B).$$

3. The base BC of an isosceles triangle ABC makes an angle θ with a straight line BX. If $\angle BAC = 2\phi$, prove, by projecting the sides of the triangle on BX, that

$$2\sin\phi\cos\theta = \sin(\phi + \theta) + \sin(\phi - \theta).$$

4. Prove that, if β and γ are positive angles less than two right angles such that

$$\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + \beta + \gamma) = 0,$$

and $$\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + \beta + \gamma) = 0,$$

then $\beta = \gamma = \frac{2}{3}\pi$.

5. P and Q are points in the co-ordinate plane such that

$$\angle XOP = 132°\ 14';\ OP = 5 \text{ units};$$
$$\angle OPQ = -15°\ 40';\ PQ = 12 \text{ units}.$$

Find the length of OQ and the angle which OQ makes with OX.

Ans. 7·31, $-74°\ 4'$.

6. If OA = 10, $\angle XOA = -32°$, AB = 20, $\angle OAB = 54°$, find the co-ordinates of B, the length of OB, and the angle XOB.

Ans. $-10·06$, $-12·79$, 16·27, $-128°\ 11'$.

7. If OA = 7, $\angle XOA = -118°$, AB = 10, $\angle OAB = 22°$, calculate the co-ordinates of B, the angle XOB, and the length of OB.

Ans. $-2·24$, 3·76, $120°\ 46'$, 4·38.

8. From the following data, OA = 15, $\angle XOA = 123°$, AB = 19, $\angle OAB = 72°$, BC = 32, $\angle ABC = 24°$, CD = 25, $\angle BCD = -48°$, calculate the co-ordinates of D, the angle XOD, and the length of OD.

Ans. 10·01, $-6·55$, $-33°\ 13'$, 11·96.

9. If OA = 10, AB = 20, BC = 5, $\angle XOA = 140°$, $\angle OAB = 55°$, $\angle ABC = 82°$, find (i) the projections of OC on OX and OY, (ii) the angle XOC and the length of OC.

Ans. (i) 12·27, 6·64, (ii) $28°\ 26'$, 13·95.

10. The data of a traverse are as follows: $\angle XOA = 19° 39'$, $\angle OAB = -47° \ 23'$, $\angle ABC = -34° \ 24'$; $OA = 105$, $AB = 43$, $BC = 197$. Determine OC and $\angle XOC$.

Ans. $194, -37° 51'$.

11. ABCD is a quadrilateral with $AB = 9$, $BC = 27$, $CD = 15$, $\angle ABC = -135° \ 15'$, $\angle BCD = -13° \ 12'$. Calculate the length of AD and the angle BAD.

Ans. $19, 35° 56'$.

12. Lanark is 11 miles due east of Strathaven. Bathgate lies 18° east of north from Lanark, and 16 miles from it. Wishaw lies 40° south of west from Bathgate, and 14 miles from it. Find the direction and the distance of Wishaw from Strathaven.

Ans. 40° 1′ east of north, 8·12 miles.

13. A man surveying a road moves from A, a distance 6·7 chains in a direction 56° east of south, then 5·3 chains in a direction 42° east of north, then 5·7 chains in a direction 30° west of north to B. Find how far and in what direction B is from A.

Ans. 8·085 chains, 39° 22′ N. of E.

14. OABCO is a quadrilateral, the vertices being named in counter-clockwise order, and O is the origin. Given that $OA = 8$, $AB = 11$, $BC = 5$ units, $\angle XOA = +27°$, and that the internal angles of the quadrilateral at A and B are numerically equal to 76° and 98° respectively, calculate the length of OC and the magnitude of the angle XOC.

Ans. 10·16 units, 114° 56′.

15. A quadrilateral ABCD represents the section of a railway cutting, normal to the line of rails. AB is the horizontal bottom, 30 feet long; BC is 40 feet long, DA 25 feet long, and $\angle CBA = 145°$, $\angle BAD = 150°$. Calculate (i) the angle of slope of the hillside DC with the horizontal, (ii) the distance DC.

Ans. 7° 3′, 85 feet.

16. A circle touches the x-axis at O, and intersects the y-axis above the origin at B. A is a point on that part of the circle which lies to the right of OB, and the tangents at A and B meet at T. If the angle AOB is θ, find the angles which the directed lines OA, AT and TB make with OX. The lengths of OA, AT and OB are c, t and d respectively; show, by projecting on OX and OY, that

(i) $c \sin\theta - t(1 + \cos 2\theta) = 0$,
(ii) $c \cos\theta + t \sin 2\theta = d$.

Ans. $\frac{1}{2}\pi - \theta, \ \pi - 2\theta, \ \pi$.

CHAPTER VI

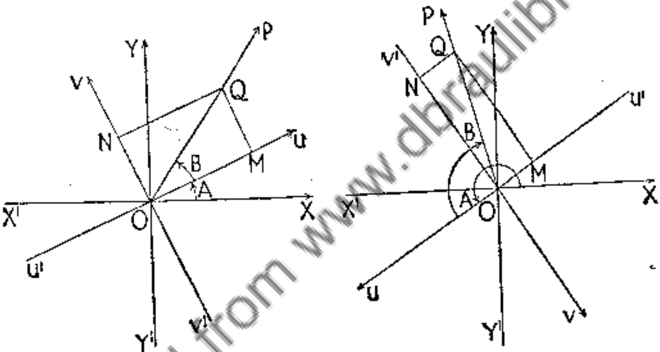
ADDITION THEOREMS

§ 1. The Addition Theorems

THE theorems on projection established in the previous chapter will now be employed to give a proof* of the formula

$$\cos(A + B) = \cos A \cos B - \sin A \sin B, \quad (1)$$

valid for all values of A and B. Two figures (1(*a*), (*b*)) are



(*a*) FIG. 1. (*b*)

given, but the reader would be well advised to draw diagrams for other values of A and B.

In the *xy*-plane let angles XOU and UOP be equal to A and B respectively. Take new rectangular axes U′OU, V′OV such that $\angle$UOV is a positive right angle. From Q, a point on OP, draw QM and QN perpendicular to U′OU and V′OV respectively. Then

$$\angle XOP = \angle XOU + \angle UOP = A + B,$$

so that, if $OQ = r$,

$$\text{Proj. of OQ on X'X} = r\cos(A + B).$$

* An alternative proof will be found in Chap. IX, § 8.

But $\quad$ Proj. of OQ = Proj. of OM + Proj. of MQ
$\qquad\qquad$ = Proj. of OM + Proj. of ON,

by Chap. V, § 2, Theorems II and III.

Now OM and ON are segments of the directed lines U′OU and V′OV which make angles A and A + 90° respectively with OX. Hence (Chap. V, § 2, Theorem IV)

$$\begin{aligned}\text{Proj. of OQ on X}'\text{X} &= \text{OM}\cos A + \text{ON}\cos(A + 90^\circ)\\ &= (\text{Proj. of OQ on U}'\text{U}) \times \cos A\\ &\qquad + (\text{Proj. of OQ on V}'\text{V}) \times \cos(A + 90^\circ)\\ &= r\cos B\cos A + r\sin B\cos(A + 90^\circ).\end{aligned}$$

Thus $\quad \cos(A + B) = \cos A\cos B - \sin A\sin B. \quad . \quad (1)$

The addition theorem for the sine

$$\sin(A + B) = \sin A\cos B + \cos A\sin B \quad . \quad (2)$$

can be proved in just the same way by projecting on the y-axis instead of on the x-axis. Thus

$$\begin{aligned}r\sin(A + B) &= \text{Proj. of OQ on Y}'\text{Y}\\ &= \text{Proj. of OM on Y}'\text{Y} + \text{Proj. of MQ on Y}'\text{Y}\\ &= \text{Proj. of OM on Y}'\text{Y} + \text{Proj. of ON on Y}'\text{Y}\\ &= \text{OM}\sin A + \text{ON}\sin(A + 90^\circ)\\ &= (\text{Proj. of OQ on U}'\text{U}) \times \sin A\\ &\qquad + (\text{Proj. of OQ on V}'\text{V}) \times \sin(A + 90^\circ)\\ &= r\cos B\sin A + r\sin B\sin(A + 90^\circ),\end{aligned}$$

from which (2) follows.

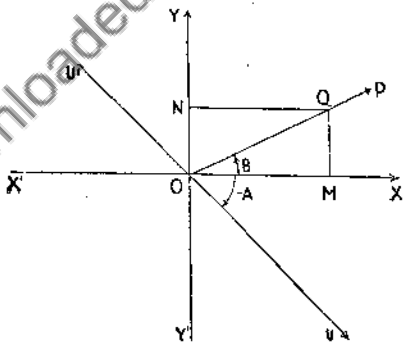


FIG. 2.

Note.—Formulæ (1) and (2) can be deduced from each other by putting B + 90° in place of B.

Modified Proof for the Cosine.—The following modification of the proof in the case of the cosine is due to Dr. James Hyslop.

Let U′OU and OP (Fig. 2) be directed lines such that $\angle XOU = -A$ and $\angle XOP = B$. Then $\angle UOX = A$, so that $\angle UOP = \angle UOX + \angle XOP = A + B$,
and $\angle UOY = \angle UOX + \angle XOY = A + 90°$.

From Q, a point on OP, draw QM and QN perpendicular to X′X and Y′Y respectively. Then, if $OQ = r$,

$$\begin{aligned}
&\text{Proj. of OQ on U′U} = r\cos(A+B)\\
&= \text{Proj. of OM on U′U} + \text{Proj. of MQ on U′U}\\
&= \text{Proj. of OM on U′U} + \text{Proj. of ON on U′U}\\
&= OM\cos A + ON\cos(A+90°)\\
&= (\text{Proj. of OQ on X′X}) \times \cos A\\
&\qquad + (\text{Proj. of OQ on Y′Y}) \times \cos(A+90°)\\
&= r\cos B\cos A + r\sin B\cos(A+90°)
\end{aligned}$$

so that

$$\cos(A+B) = \cos A\cos B - \sin A\sin B.$$

Addition Formula for the Tangent.—From (1) and (2) it follows that

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B};$$

hence, on dividing numerator and denominator by $\cos A \cos B$, we have

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}. \qquad (3)$$

Difference Formulæ.—By replacing B by $-$B in (1), (2) and (3), it can be deduced that

$$\cos(A-B) = \cos A\cos B + \sin A\sin B, \qquad (4)$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B, \qquad (5)$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}. \qquad (6)$$

Example 1.—With the aid of the equality $15° = 45° - 30°$, prove that

(i) $\sin 15° = \frac{\sqrt{2}}{4}(\sqrt{3} - 1)$, (ii) $\cos 15° = \frac{\sqrt{2}}{4}(\sqrt{3} + 1)$,

(iii) $\tan 15° = 2 - \sqrt{3}$.

Example 2.—If $A + B + C = 0$, show that

$$\tan A + \tan B + \tan C = \tan A\tan B\tan C.$$

Example 3.—Show that

(i) $\cos\theta + \cos(\theta + 120°) + \cos(\theta - 120°) = 0$,

(ii) $\sin(A + B) + \cos(A - B)$
$$= (\sin A + \cos A)(\sin B + \cos B),$$

(iii) $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$,

(iv) $\cos(45° - A)\cos(45° - B)$
$$- \sin(45° - A)\sin(45° - B) = \sin(A + B).$$

Example 4.—Prove that

(i) $\cot(A + B) = \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$,

(ii) $\cot(A - B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$,

(iii) $\tan(A + 45°) = \dfrac{1 + \tan A}{1 - \tan A}$,

(iv) $\tan(A - 45°) = \dfrac{\tan A - 1}{\tan A + 1}$,

(v) $\tan 75° = 2 + \sqrt{3}$.

Duplication Formulæ.—When B is equal to A, formulæ (1), (2), (3) become

$$\cos 2A = \cos^2 A - \sin^2 A, \qquad (7)$$
$$\sin 2A = 2\sin A\cos A, \qquad (8)$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}. \qquad (9)$$

From (7), by employing the identity $\cos^2 A + \sin^2 A = 1$, it can be deduced that

$$\cos 2A = 2\cos^2 A - 1, \qquad (10)$$

and
$$\cos 2A = 1 - 2\sin^2 A. \qquad (11)$$

It is frequently found useful to write (10) and (11) in the forms

$$\cos^2 A = \frac{1 + \cos 2A}{2}, \qquad (12)$$

and
$$\sin^2 A = \frac{1 - \cos 2A}{2}. \qquad (13)$$

If cos 2A is known, (12) and (13) may be employed to obtain cos A and sin A. The sign of the cosine or sine is of course determined by the quadrant in which the angle lies.

Example 5.—By means of (12) and (13), prove that

(i) $\cos 15^\circ = \frac{\sqrt{2}}{4}(\sqrt{3}+1)$, (ii) $\sin 15^\circ = \frac{\sqrt{2}}{4}(\sqrt{3}-1)$,

(iii) $\cos 22\frac{1}{2}^\circ = \frac{\sqrt{(2+\sqrt{2})}}{2}$, (iv) $\sin 22\frac{1}{2}^\circ = \frac{\sqrt{(2-\sqrt{2})}}{2}$.

Example 6.—Show that

(i) $\tan A = \frac{\sin 2A}{1+\cos 2A}$, (ii) $\cot^2 A = \frac{1+\cos 2A}{1-\cos 2A}$.

Example 7.—Solve the equation

$$\cos 2\theta + \sin\theta + 1 = 0.$$

Using (11), we can write this as a quadratic in $\sin\theta$

$$2\sin^2\theta - \sin\theta - 2 = 0.$$

This gives

$$\sin\theta = \frac{1 \pm \sqrt{(17)}}{4} = 1{\cdot}28078 \text{ or } -{\cdot}78078,$$

and from the second value it follows that

$$\theta = n\,.\,180^\circ - (-1)^n 51^\circ\ 20'.$$

Example 8.—Solve the equation

$$\cos 2\theta + \cos\theta - 1 = 0.$$

Ans. $\theta = n\,.\,360^\circ \pm 38^\circ\ 40'$.

[By means of (10) express the equation as a quadratic in $\cos\theta$.]

Expressions for $\cos 2A$ *and* $\sin 2A$ *in terms of* $\tan A$.—Formulæ (7) and (8) may be written

$$\cos 2A = \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A}, \quad \sin 2A = \frac{2\sin A\cos A}{\cos^2 A + \sin^2 A}.$$

In each case divide numerator and denominator by $\cos^2 A$, and so obtain the important formulæ

$$\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}, \quad . \quad . \quad . \quad (14)$$

$$\sin 2A = \frac{2\tan A}{1+\tan^2 A} \quad . \quad . \quad . \quad (15)$$

Example 9.—If $\tan\theta = \frac{\tan\alpha + \tan\beta}{1+\tan\alpha\tan\beta}$,

show that $$\sin 2\theta = \frac{\sin 2\alpha + \sin 2\beta}{1+\sin 2\alpha \sin 2\beta}.$$

Triplication Formulæ.—Since $\cos 3A = \cos (2A + A)$,

$$\cos 3A = \cos 2A \cos A - \sin 2A \sin A$$
$$= (2\cos^2 A - 1)\cos A - 2\sin A \cos A \,.\, \sin A$$
$$= (2\cos^2 A - 1)\cos A - 2\cos A\,(1 - \cos^2 A),$$

and therefore $$\cos 3A = 4\cos^3 A - 3\cos A \quad . \quad . \quad (16)$$

Similarly

$$\sin 3A = \sin 2A \cos A + \cos 2A \sin A$$
$$= 2\sin A \cos A \,.\, \cos A + (1 - 2\sin^2 A)\sin A$$
$$= 2\sin A\,(1 - \sin^2 A) + (1 - 2\sin^2 A)\sin A,$$

so that $$\sin 3A = 3\sin A - 4\sin^3 A \quad . \quad . \quad (17)$$

Example 10.—Find the value of $\sin 18°$.

We have

$$\cos (3 \times 18°) = \cos 54° = \sin 36° = \sin (2 \times 18°).$$

Hence, $4\cos^3 18° - 3\cos 18° = 2\sin 18° \cos 18°$,

or $\cos 18°\,(4\sin^2 18° + 2\sin 18° - 1) = 0$.

Thus, $$\sin 18° = \frac{-1 \pm \sqrt{5}}{4}.$$

But $\sin 18°$ is positive ; therefore

$$\sin 18° = \frac{\sqrt{5} - 1}{4}.$$

Example 11.—Show that $\cos 36° = (\sqrt{5} + 1)/4$.

[Use the equation $\sin (3 \times 36°) = \sin (2 \times 36°)$, or deduce the result from *Example* 10.]

The Inverse Tangent.—The addition and difference formulæ for the inverse tangent are

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x + y}{1 - xy} \quad . \quad . \quad (18)$$

and $$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x - y}{1 + xy} \quad . \quad . \quad (19)$$

These can be deduced from (3) and (6) as follows : let

$$\alpha = \tan^{-1} x, \quad \beta = \tan^{-1} y\,;$$

then $$\tan (\alpha + \beta) = \frac{x + y}{1 - xy}, \quad \tan (\alpha - \beta) = \frac{x - y}{1 + xy},$$

so that

$$\alpha + \beta = \tan^{-1}\frac{x+y}{1-xy}, \quad \alpha - \beta = \tan^{-1}\frac{x-y}{1+xy},$$

from which (18) and (19) follow.

Example 12.—Show that

$$\tan^{-1}\tfrac{1}{7} + \tan^{-1}\tfrac{5}{12} + 2\tan^{-1}\tfrac{1}{8} = \frac{\pi}{4}.$$

Example 13.—Prove that

(i) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\{x\sqrt{(1-y^2)} + y\sqrt{(1-x^2)}\}$,
(ii) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\{x\sqrt{(1-y^2)} - y\sqrt{(1-x^2)}\}$,
(iii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\{xy - \sqrt{(1-x^2)}\sqrt{(1-y^2)}\}$,
(iv) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\{xy + \sqrt{(1-x^2)}\sqrt{(1-y^2)}\}$.

§ 2. Addition Theorems for any Number of Angles

Let T_r ($r = 1, 2, \ldots, n$) denote the sum of the products of $t_1, t_2, \ldots, t_n$ taken r at a time, where $t_s = \tan\theta_s$; then

$$\cos(\theta_1 + \theta_2 + \ldots + \theta_n) = \cos\theta_1\cos\theta_2\ldots\cos\theta_n(1 - T_2 + T_4 - \ldots), \quad (20)$$

$$\sin(\theta_1 + \theta_2 + \ldots + \theta_n) = \cos\theta_1\cos\theta_2\ldots\cos\theta_n(T_1 - T_3 + T_5 - \ldots), \quad (21)$$

$$\tan(\theta_1 + \theta_2 + \ldots + \theta_n) = \frac{T_1 - T_3 + T_5 - \ldots}{1 - T_2 + T_4 - \ldots}. \quad (22)$$

Formula (22) follows immediately from (20) and (21). These will be established by the method of mathematical induction.

Assume that (20) and (21) are true for the sum of n angles: then

$$\begin{aligned}\cos(\theta_1 + \theta_2 + \ldots + \theta_n + \theta_{n+1}) &= \cos(\theta_1 + \theta_2 + \ldots + \theta_n)\cos\theta_{n+1} \\ &\quad - \sin(\theta_1 + \theta_2 + \ldots + \theta_n)\sin\theta_{n+1}\end{aligned}$$

$$= \cos\theta_1\cos\theta_2\ldots\cos\theta_{n+1}\begin{Bmatrix}(1 - T_2 + T_4 - \ldots) \\ -(T_1 - T_3 + T_5 - \ldots)t_{n+1}\end{Bmatrix}$$

$$= \cos\theta_1\cos\theta_2\ldots\cos\theta_{n+1}(1 - T_2' + T_4' - \ldots),$$

where T_r' ($r = 1, 2, \ldots, n+1$) denotes the sum of the products of $t_1, t_2, \ldots, t_{n+1}$ taken r at a time.

Again

$$\sin(\theta_1 + \theta_2 + \ldots + \theta_n + \theta_{n+1})$$
$$= \sin(\theta_1 + \theta_2 + \ldots + \theta_n)\cos\theta_{n+1}$$
$$+ \cos(\theta_1 + \theta_2 + \ldots + \theta_n)\sin\theta_{n+1}$$
$$= \cos\theta_1 \cos\theta_2 \ldots \cos\theta_{n+1}\left\{\begin{matrix}(T_1 - T_3 + T_5 - \ldots) \\ + (1 - T_2 + T_4 - \ldots)t_{n+1}\end{matrix}\right\}$$
$$= \cos\theta_1 \cos\theta_2 \ldots \cos\theta_{n+1}(T_1' - T_3' + T_5' - \ldots).$$

Thus, if (20) and (21) are true for the sum of n angles, they are also true for the sum of $n + 1$ angles. But these formulæ are true when n is equal to 1 or 2 : hence they hold for all values of n.

COROLLARY.—Just as (18) was deduced from (3), so from (22) can be deduced the formula

$$\tan^{-1} t_1 + \tan^{-1} t_2 + \ldots + \tan^{-1} t_n = \tan^{-1}\frac{T_1 - T_3 + T_5 - \ldots}{1 - T_2 + T_4 - \ldots}. \quad (23)$$

Example.—If $A + B + C = 90°$, prove that
$$\tan A \tan B + \tan B \tan C + \tan C \tan A = 1.$$

Multiplication Formulæ.—If in (20), (21) and (22) the angles are all put equal to θ, these formulæ become

$$\cos n\theta = \cos^n\theta\,(1 - {}^nC_2 \tan^2\theta + {}^nC_4 \tan^4\theta - \ldots), \quad (24)$$
$$\sin n\theta = \cos^n\theta\,({}^nC_1 \tan\theta - {}^nC_3 \tan^3\theta + {}^nC_5 \tan^5\theta - \ldots), \quad (25)$$
$$\tan n\theta = \frac{{}^nC_1 \tan\theta - {}^nC_3 \tan^3\theta + {}^nC_5 \tan^5\theta - \ldots}{1 - {}^nC_2 \tan^2\theta + {}^nC_4 \tan^4\theta - {}^nC_6 \tan^6\theta + \ldots}, \quad (26)$$

where, of course, n is a positive integer.

EXAMPLES VI

1. Establish the identities

(i) $\cos A \sin(B - C) + \cos B \sin(C - A) + \cos C \sin(A - B) = 0,$

(ii) $\cos(36° - A)\cos(36° + A) + \cos(54° - A)\cos(54° + A) = \cos 2A,$

(iii) $\sin 70° \cos\theta = \sin 61° \cos(49° + \theta) + \sin 49° \sin(29° + \theta).$

2. If A, B, C are the angles of a triangle, show that

$$\cos B \cos \theta - \sin A \sin (\theta + C) - \cos C \cos (\theta + B + C) = 0.$$

3. If $A + B + C = 180^\circ$ and $\sin A + \sin B \cos C = 0$, prove that $2 \tan B + \tan C = 0$.

4. If $\sin A \sin (B - C) = \cos A \cos (B + C)$,
show that $\sin C \sin (B - A) = \cos C \cos (B + A)$.

5. If $\cos \beta \sin (\alpha - \theta) = \sin \beta \cos (\alpha + \theta)$,
show that $\cos \theta \sin (\alpha - \beta) = \sin \theta \cos (\alpha + \beta)$.

6. If $\cos \theta = \cos A \cos (A - \theta)$, show that $\theta = A + n\pi$, where n is integral.

7. Show that

(i) $\sin (A + \theta) \cos B + \cos A \sin (B - \theta) = \sin (A + B) \cos \theta$,

(ii) $\sin (A + B) \cos (C - D) + \sin (A - B) \cos (C + D) = \sin (A + C) \cos (B - D) + \sin (A - C) \cos (B + D)$,

(iii) $\sin^2 A + \sin^2 B + \cos^2 (A + B) + 2 \sin A \sin B \cos (A + B) = 1$,

(iv) $\cos (A - B)\{\sin (A + B) + 1\} = (\sin A + \cos B)(\cos A + \sin B)$,

(v) $\sin (A + C) \sin (B + D) - \sin (A + D) \sin (B + C) = \sin (A - B) \sin (D - C)$,

(vi) $\{\sin (A + B) + \cos (A + B)\} \times \{\sin (A - B) + \cos (A - B)\} = \sin 2A + \cos 2B$.

8. Establish the identity

$$\frac{\sin A \sin (B - C) + \sin B \sin (C - A)}{\cos A \sin (B - C) + \cos B \sin (C - A)} = \tan C.$$

9. Show that

$$\sin^2 (\theta + \alpha) + \sin^2 (\theta + \beta) - 2 \sin (\theta + \alpha) \sin (\theta + \beta) \cos (\alpha - \beta) = \sin^2 (\alpha - \beta).$$

10. If $$\frac{\cos (\theta_1 - \theta_2)}{\cos (\theta_1 + \theta_2)} + \frac{\cos (\theta_3 + \theta_4)}{\cos (\theta_3 - \theta_4)} = 0,$$
prove that $\tan \theta_1 \tan \theta_2 \tan \theta_3 \tan \theta_4 = -1$.

11. If $A + B + C = 90^\circ$, show that

$$\frac{\sin^2 B + \sin^2 C - \cos^2 A}{\cos^2 B + \cos^2 C - \cos^2 A} = -\tan B \tan C.$$

12. If $u_n = \sin n\theta \,.\, (\sec \theta)^n$, $v_n = \cos n\theta \,.\, (\sec \theta)^n$, where $n = 0, 1, 2, 3, \ldots$, prove that

$$v_n - v_{n-1} = -u_{n-1} \,.\, \tan \theta.$$

Deduce that

$$u_1 + u_2 + \ldots + u_n = \cot \theta \,.\, (\sec \theta)^{n+1}\{(\cos \theta)^{n+1} - \cos (n + 1)\theta\}.$$

13. If $2\cos(x+\theta)\cos(x-\theta)=1$, show that

$$\tan^2 x = \frac{1-\tan^2\theta}{1+3\tan^2\theta}.$$

14. If the angle α is divided into two parts, θ and ϕ, so that $\sin\theta = \kappa\sin\phi$, prove that

$$\text{(i)}\ \tan\theta = \frac{\kappa\sin\alpha}{1+\kappa\cos\alpha}, \quad \text{(ii)}\ \tan\phi = \frac{\sin\alpha}{\kappa+\cos\alpha}.$$

Hence, find two angles whose sum is 60°, such that the sine of one is twice the sine of the other.

Ans. 40° 54′, 19° 6′.

15. If $\cos x = \sec(\alpha-\beta)\cos(\alpha+\beta-x)$, show that

$$\tan x = \frac{2\sin\alpha\sin\beta}{\sin(\alpha+\beta)}.$$

16. If $\sin(\theta+\phi+x)\sec x = \tan\theta + \tan\phi$,
show that $\tan x = \tan(\theta+\phi)(\sec\theta\sec\phi - 1)$.

17. Show that

$$\tan A\tan(A-B)\{\cot A + \tan B\} = \tan A - \tan B.$$

18. If $\cos^2 B\tan(A+\theta) = \sin^2 B\cot(A-\theta)$,
show that $\tan^2\theta = \tan(A+B)\tan(A-B)$.

19. If the angle A is divided into two parts, α and β, so that $\tan\alpha = k\tan A$, show that

$$\tan\beta = \frac{(1-k)\sin A\cos A}{\cos^2 A + k\sin^2 A}.$$

20. If $\cot\theta = \cos(x+y)$ and $\cot\phi = \cos(x-y)$, prove that

$$\tan(\theta-\phi) = \frac{2\sin x\sin y}{\cos^2 x + \cos^2 y}.$$

21. If $A + B = 45°$, prove that

$$(1+\tan A)(1+\tan B) = 2.$$

Deduce that $\tan 22\frac{1}{2}° = \sqrt{2}-1$, and evaluate $\tan 7\frac{1}{2}°$.

Ans. $(\sqrt{2}-1)(\sqrt{3}-\sqrt{2})$.

22. Without using tables, find the value of

$$(1-\tan 15°)/(1+\tan 15°).$$

Ans. $1/\sqrt{3}$.

23. Show that

(i) $\cot A - \operatorname{cosec} 2A = \cot 2A$,

(ii) $\dfrac{\sin(A+B)+\cos(A-B)}{\sin(A-B)+\cos(A+B)} = \sec 2B + \tan 2B$,

(iii) $2\cos^2 A\sin(A+B) - \sin 2A\cos(A+B)$
$= \sin 2B\cos(A-B) - 2\sin^2 B\sin(A-B)$,

(iv) $\sin^2 (A + B) - \sin^2 (A - B) = \sin 2A \sin 2B$.

(v) $1 - \tan^2 \frac{1}{2}A \cdot \tan^2 A = 2 \sec A - \sec^2 A$,

(vi) $\cot^2 A \cdot \tan^2 2A - 1 = (1 + 2 \cos 2A) \sec^2 2A$.

24. Show that $\sin \theta$ is a factor of

$$\sin \theta + \sin \phi - \cos \theta \cdot \sin (\theta + \phi),$$

and find the other factors.

Ans. $2 \sin^2 \frac{1}{2}(\theta + \phi)$.

25. If $\sin x = a \sin (2y - x)$, show that

$$\tan x = \frac{2a \tan y}{(a + 1) - (a - 1) \tan^2 y}.$$

26. If $$\frac{\cos (x - \theta)}{a} = \frac{\cos (x + \theta)}{b} = \frac{\sin 2\theta}{c},$$

prove that

$$\text{(i) } \frac{\sin \theta}{c} = \frac{\cos x}{a + b}, \quad \text{(ii) } \frac{\cos \theta}{c} = \frac{\sin x}{a - b},$$

$$\text{(iii) } \cos 2\theta = \frac{a^2 + b^2 - c^2}{2ab}.$$

27. If $x = r \sin (\theta - \alpha)$ and $y = r \sin (\theta + \alpha)$, prove that

$$x^2 - 2xy \cos 2\alpha + y^2 = r^2 \sin^2 2\alpha.$$

28. Show that, if $\sin B = \sin A - \sin C$,

$$\cos 2A - \cos 2B + \cos 2C = 1 - 4 \sin A \sin C.$$

29. If $\tan \theta = \cos \alpha \tan \phi$, prove that

$$\tan (\phi - \theta) = \frac{\tan^2 \frac{1}{2}\alpha \sin 2\phi}{1 + \tan^2 \frac{1}{2}\alpha \cos 2\phi}.$$

30. Show that

$$\frac{\tan \theta + \sec \theta - 1}{\tan \theta - \sec \theta + 1} = \frac{1 + \tan \frac{1}{2}\theta}{1 - \tan \frac{1}{2}\theta}.$$

31. If $\sec x = \cot A \cot B$, prove that

$$\tan^2 \tfrac{1}{2}x = \cos (A + B) \sec (A - B).$$

32. If $\tan \alpha$, $\tan \beta$, $\tan \gamma$ are in geometrical progression, show that

$$\cos 2\beta = \cos (\alpha + \gamma) \sec (\alpha - \gamma).$$

33. Prove that

$$2 \operatorname{cosec} 4\theta - \sec 2\theta = \frac{1 - \tan \theta}{1 + \tan \theta} \operatorname{cosec} 2\theta.$$

34. If $$\tan \theta = \sqrt{\left(\frac{a - b}{a + b}\right)} \cdot \tan \tfrac{1}{2}A,$$

and $$\cos \phi = (b + a \cos A)/(a + b \cos A),$$

the angles θ and ϕ being acute, prove that $\phi = 2\theta$.

35. If A is an angle between 270° and 360° whose tangent is $-24/7$, calculate as vulgar fractions the sine and the cosine of $\frac{1}{2}$A.

Ans. $\frac{3}{5}$, $-\frac{4}{5}$.

36. Find solutions between 0° and 360° for the following equations.

(i) $2 \sin 2\theta = \cos \theta$.

Ans. 90°, 270°, 14° 29′, 165° 31′.

(ii) $\cos 2\theta - 4 \sin \theta + 3 = 0$.

Ans. 47° 4′, 132° 56′.

(iii) $\cos 2\theta + 2 \cos \theta = 1$.

Ans. 51° 50′, 308° 10′.

(iv) $4 \cos 2\theta = 2 \sin \theta + 1$.

Ans. 30°, 150°, 228° 35′, 311° 25′.

(v) $16(\cos 2\theta + \sin \theta)(\cos 2\theta - \sin \theta) = 7$.

Ans. 20° 42½′, 200° 42½′, 159° 17½′, 339° 17½′.

(vi) $\sec \theta = 3 \cot \theta - 7 \cot 2\theta$.

Ans. 30°, 150°, 194° 29′, 345° 31′.

(vii) $\sin 3\theta = 2 \sin \theta$.

Ans. 0°, 180°, 360°, 30°, 150°, 210°, 330°.

(viii) $\cos 3\theta - 2 = 2 \cos 2\theta$.

Ans. 90°, 270°, 120°, 240°.

(ix) $2 \cos 2\theta - \sin 3\theta = 2$.

Ans. 0°, 180°, 360°, 210°, 330°.

37. Show that

(i) $\sin^3 A \sin 3A + \cos^3 A \cos 3A = \cos^3 2A$,

(ii) $\cos 3A = \sin 3A + (\cos A + \sin A)(1 - 2 \sin 2A)$,

(iii) $4 \cos A \cos (120° - A) \cos (120° + A) = \cos 3A$,

(iv) $\frac{1}{3} \sin 3A \cos^3 A + \frac{1}{3} \cos 3A \sin^3 A = \frac{1}{4} \sin 4A$.

38. Prove that

(i) $\cos \theta = \dfrac{\cos 3\theta}{2 \cos 2\theta - 1}$, (ii) $\sin \theta = \dfrac{\sin 3\theta}{2 \cos 2\theta + 1}$,

and hence find the values of cos 15° and sin 15°.

Ans. $(\sqrt{3} + 1)/(2\sqrt{2})$, $(\sqrt{3} - 1)/(2\sqrt{2})$.

39. Find all the solutions between 0° and 360° of the equation

$$\cos x \cos 3x = -\tfrac{1}{4}.$$

Ans. 36°, 72°, 108°, 144°, 216°, 252°, 288°, 324°.

40. Solve the equation

$$\cos 3\theta \cos 2\theta - \cos \theta = 0,$$

for values of θ between 0° and 360°.

Ans. 0°, 90°, 180°, 270°, 360°, 0°, 60°, 120°, 180°, 240°, 300°, 360°.

41. Verify that $\cos 20°$, $\cos 100°$ and $\cos 140°$ are solutions of the cubic equation $8x^3 - 6x - 1 = 0$, and hence show that $\cos 20° \cos 100° + \cos 100° \cos 140° + \cos 140° \cos 20° = -\frac{3}{4}$.

42. Verify that $\sin 10°$, $\sin 50°$ and $\sin 250°$ are solutions of the equation $8x^3 - 6x + 1 = 0$, and hence show that

$$\sin 10° \sin 50° + \sin 50° \sin 250° + \sin 250° \sin 10° = -\tfrac{3}{4}.$$

43. If $\sin^3 x \sin \frac{1}{2}x = \cos^3 x \cos \frac{1}{2}x$, prove that

$$3 \cos \tfrac{3}{2}x + \cos \tfrac{5}{2}x = 0.$$

44. Prove that

$$\frac{\sin 3\theta - \sin \theta \sin^2 2\theta}{\sin \theta + \sin 2\theta \cos \theta} = \cos 2\theta.$$

45. In a circle of radius r inches, chords of length a and b inches subtend angles θ and 3θ respectively at the circumference. Show that

$$r = a\sqrt{\left(\frac{a}{3a-b}\right)}.$$

46. Show that the quadratic

$$x^2 \sin A \sin 3A - 2x \cos A \sin 2A + 1$$

vanishes for $x = \operatorname{cosec} A$, and hence find the factors of the expression.

Deduce the solutions of the equation

$$4(\sin \theta \sin 3\theta - \cos \theta \sin 2\theta) + 1 = 0,$$

giving all roots between 0° and 360°.

Ans. $(x \sin A - 1)(x \sin 3A - 1)$; 30°, 150°, 10°, 50° 130°, 170°, 250°, 290°.

47. Show that $\tan 3A(1 - 3 \tan^2 A) = 3 \tan A - \tan^3 A$, and employ this relation to solve the equation

$$x^3 - (5{\cdot}457)x^2 - 3x + 1{\cdot}819 = 0.$$

Ans. $\tan (20° \ 24') = {\cdot}37190$, $\tan (80° \ 24') = 5{\cdot}91236$, $\tan (140° \ 24') = -{\cdot}82727$.

48. If $\tan 3A/\tan A = k$, show that $\sin 3A/\sin A = 2k/(k-1)$, and that k cannot lie between 3 and $\frac{1}{3}$.

49. Show that

$$\cos (m+1)A = 2 \cos A \cos mA - \cos (m-1)A,$$

and hence express $\cos 4A$ in a series of powers of $\cos A$.

Ans. $8 \cos^4 A - 8 \cos^2 A + 1$.

50. PQR is an isosceles triangle whose equal sides PQ and PR are at right angles; S and T are points on PQ such that QS = 6SP and QT = 2TP. Prove that

$$2\angle PRT + \angle PRS = \angle PRQ.$$

51. Show that

(i) $4\{\cos^4(\theta + 45^\circ) + \cos^4(\theta - 45^\circ)\} = 3 - \cos 4\theta$,
(ii) $8(\sin^6\theta + \cos^6\theta) = 5 + 3\cos 4\theta$.

52. Show that, if $A + B + C = 180^\circ$,

$$\cot \tfrac{1}{2}A + \cot \tfrac{1}{2}B + \cot \tfrac{1}{2}C = \cot \tfrac{1}{2}A \cot \tfrac{1}{2}B \cot \tfrac{1}{2}C.$$

53. Prove that, if α, β, γ, δ are four non-coterminal roots of the equation

$$a \sec\theta + b \operatorname{cosec}\theta + c = 0,$$

then $\alpha + \beta + \gamma + \delta = (2n + 1)\pi$, where n is an integer.

54. If α, β, γ, δ are four non-coterminal values of θ satisfying the equation

$$\cos(2\theta - \omega) = a\cos\theta + b,$$

show that $$\alpha + \beta + \gamma + \delta = 2(n\pi + \omega).$$

55. If α, β, γ, δ are the smallest positive angles in ascending order of magnitude which have their sines equal to the positive quantity k, show that

$$4\sin\tfrac{1}{2}\alpha + 3\sin\tfrac{1}{2}\beta + 2\sin\tfrac{1}{2}\gamma + \sin\tfrac{1}{2}\delta = 2\sqrt{(1 + k)}.$$

56. With centre a point A on a given circle an arc BD is described within the circle to meet it in B and D. If θ is the number of radians in $\angle$BAD, show that the arc BD bisects the given circle if $\sin\theta - \theta\cos\theta = \frac{1}{2}\pi$. Verify that this condition is approximately satisfied when $\angle BAD = 109^\circ\ 11'$.

CHAPTER VII

TRANSFORMATIONS OF PRODUCTS AND SUMS

§ 1. Expressions for Products as Sums and Differences

THE formulæ

$$2 \sin A \cos B = \sin (A + B) + \sin (A - B), \quad . \quad (1)$$
$$2 \cos A \sin B = \sin (A + B) - \sin (A - B), \quad . \quad (2)$$
$$2 \cos A \cos B = \cos (A + B) + \cos (A - B), \quad (3)$$
$$2 \sin A \sin B = \cos (A - B) - \cos (A + B) \quad . \quad (4)$$

can be verified by applying the formulæ (1), (2), (4), (5) of Chapter VI to the expressions on the right.

Example 1.—Prove that

$$2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} = 0.$$

Example 2.—Show that $2x + \cos (A - B)$ is a factor of

$$2x^3 - 3x^2 \cos (A - B) - 2x \cos^2 (A + B) + \sin 2A \sin 2B \cos (A - B),$$

and find the remaining factors.

Ans. $x - 2 \cos A \cos B$, $x - 2 \sin A \sin B$.

§ 2. Expressions for Sums and Differences as Products

By expressing the angles A and B in the forms

$$A = \tfrac{1}{2}(A + B) + \tfrac{1}{2}(A - B),\ B = \tfrac{1}{2}(A + B) - \tfrac{1}{2}(A - B),$$

and applying the formulæ (1), (2), (4), (5) of Chapter VI to these sums and differences, it is found that

$$\sin A + \sin B = 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B), \quad . \quad (5)$$
$$\sin A - \sin B = 2 \cos \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B), \quad . \quad (6)$$
$$\cos A + \cos B = 2 \cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B), \quad (7)$$
$$\cos A - \cos B = - 2 \sin \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B). \quad (8)$$

Example 1.—Show that

(i) $$\frac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A} = \tan 4A,$$

(ii) $$\frac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A} = \tan 3A,$$

(iii) $\dfrac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} = \dfrac{\sin 3A}{\sin 5A}$.

[In (i) apply (5) to $\sin A + \sin 7A$ and $\sin 3A + \sin 5A$, and (7) to $\cos A + \cos 7A$ and $\cos 3A + \cos 5A$.]

Example 2.—Show that

$$\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A - B)\sin(A + B).$$

[Express the *L.H.S.** in terms of $\cos 2A$ and $\cos 2B$, and apply (8); or apply the addition theorem to the *R.H.S.*†]

Example 3.—Prove that

$$\sin 10° \sin 50° + \sin 50° \sin 250° + \sin 250° \sin 10° = -\tfrac{3}{4}.$$

$$2 \times L.H.S. = \cos 40° - \cos 60° + \cos 200° - \cos 300° + \cos 240° - \cos 260°$$
$$= -\tfrac{3}{2} + \cos 40° - \cos 20° + \cos 80°.$$

But $\quad \cos 40° + \cos 80° = 2\cos 60° \cos 20° = \cos 20°.$

Hence $\quad L.H.S. = -\tfrac{3}{4}.$

Example 4.—Prove that

(i) $\sin 80° \cos 20° + \sin 45° \cos 145° + \sin 55° \cos 245° = 0$,

(ii) $\cos 32° \sin 20° + \cos 144° \cos 2° + \sin 68° \cos 56° = 0$.

§ 3. Identities involving three Angles whose Sum is two Right Angles

In the following examples it is assumed that

$$A + B + C = 180°.$$

If the angles are all positive they can be regarded as the angles of a triangle.

Example 1.—If $A + B + C = 180°$, prove that

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C.$$

An identity of this type may be established either by starting with the expression on the left-hand-side (*L.H.S.*) and proving it equal to the expression on the right-hand-side (*R.H.S.*), or else by proceeding in the reverse order. Thus, in the above example,

$$\begin{aligned} L.H.S. &= 2\sin A \cos A + 2\sin(B + C)\cos(B - C) \\ &= 2\sin A \cos(180° - B - C) + 2\sin(180° - A)\cos(B - C) \\ &= 2\sin A\{-\cos(B + C) + \cos(B - C)\} = R.H.S.; \end{aligned}$$

* Left-hand side. † Right-hand side.

or

$$R.H.S. = 2\sin A\{\cos(B - C) - \cos(B + C)\}$$
$$= \{\sin(A + B - C) + \sin(A - B + C)\}$$
$$- \{\sin(A + B + C) - \sin(B + C - A)\}$$
$$= \sin(180^\circ - 2C) + \sin(180^\circ - 2B) - \sin 180^\circ$$
$$+ \sin(180^\circ - 2A)$$
$$= L.H.S.$$

The student should work out a number of examples in both ways.

Example 2.—Prove that, if $A + B + C = 180^\circ$,

$$\cos A + \cos B - \cos C = 4\cos\tfrac{1}{2}A \,.\, \cos\tfrac{1}{2}B \,.\, \sin\tfrac{1}{2}C - 1.$$

$$L.H.S. = 2\cos\tfrac{1}{2}(A + B) \,.\, \cos\tfrac{1}{2}(A - B) - (1 - 2\sin^2\tfrac{1}{2}C)$$
$$= 2\cos(90^\circ - \tfrac{1}{2}C) \,.\, \cos\tfrac{1}{2}(A - B)$$
$$+ 2\sin\tfrac{1}{2}C \,.\, \sin\{90^\circ - \tfrac{1}{2}(A + B)\} - 1$$
$$= 2\sin\tfrac{1}{2}C\,\{\cos\tfrac{1}{2}(A - B) + \cos\tfrac{1}{2}(A + B)\} - 1$$
$$= R.H.S.\ ;$$

or

$$R.H.S. = 2\{\cos\tfrac{1}{2}(A + B) + \cos\tfrac{1}{2}(A - B)\}\sin\tfrac{1}{2}C - 1$$
$$= \sin\tfrac{1}{2}(A + B + C) - \sin\tfrac{1}{2}(A + B - C)$$
$$+ \sin\tfrac{1}{2}(C + A - B) + \sin\tfrac{1}{2}(C - A + B) - 1$$
$$= \sin 90^\circ - \sin(90^\circ - C) + \sin(90^\circ - B)$$
$$+ \sin(90^\circ - A) - 1$$
$$= L.H.S.$$

Example 3.—If $A + B + C = 180^\circ$, prove that

$$\sin^2 B + \sin^2 C - \sin^2 A = 2\sin B\sin C\cos A.$$

$$L.H.S. = \tfrac{1}{2}(1 - \cos 2B + 1 - \cos 2C) - \sin^2 A$$
$$= \cos^2 A - \cos(B + C)\cos(B - C)$$
$$= -\cos A\cos(B + C) + \cos A\cos(B - C) = R.H.S.$$

§ 4. Miscellaneous Identities

Example 1.—Establish the identity

$$1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma$$
$$= 4\sin\sigma\sin(\sigma - \alpha)\sin(\sigma - \beta)\sin(\sigma - \gamma),$$

where $2\sigma = \alpha + \beta + \gamma$.

$$L.H.S. = -\tfrac{1}{2}(\cos 2\alpha + \cos 2\beta) - \cos^2\gamma$$
$$+ \{\cos(\alpha + \beta) + \cos(\alpha - \beta)\}\cos\gamma$$
$$= -\cos(\alpha + \beta)\cos(\alpha - \beta) - \cos^2\gamma$$
$$+ \{\cos(\alpha + \beta) + \cos(\alpha - \beta)\}\cos\gamma$$
$$= \{\cos\gamma - \cos(\alpha + \beta)\}\{\cos(\alpha - \beta) - \cos\gamma\}$$
$$= 2\sin\tfrac{1}{2}(\alpha + \beta + \gamma)\sin\tfrac{1}{2}(\alpha + \beta - \gamma)$$
$$\times 2\sin\tfrac{1}{2}(\alpha - \beta + \gamma)\sin\tfrac{1}{2}(-\alpha + \beta + \gamma)$$
$$= R.H.S.$$

Example 2.—Show that

$$\sin(\beta-\gamma)+\sin(\gamma-\alpha)+\sin(\alpha-\beta)$$
$$= -4\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\gamma-\alpha)\sin\tfrac{1}{2}(\alpha-\beta).$$

$$\begin{aligned} L.H.S. &= 2\sin\tfrac{1}{2}(\beta-\gamma)\cos\tfrac{1}{2}(\beta-\gamma) \\ &\qquad + 2\sin\tfrac{1}{2}(\gamma-\beta)\cos\tfrac{1}{2}(\beta+\gamma-2\alpha) \\ &= -2\sin\tfrac{1}{2}(\beta-\gamma)\{\cos\tfrac{1}{2}(\beta+\gamma-2\alpha)-\cos\tfrac{1}{2}(\beta-\gamma)\} \\ &= -2\sin\tfrac{1}{2}(\beta-\gamma)\,.\,2\sin\tfrac{1}{2}(\beta-\alpha)\sin\tfrac{1}{2}(\alpha-\gamma) \\ &= R.H.S. \end{aligned}$$

§ 5. Solution of Equations

Formulæ (5), (6), (7), (8) are employed in the solution of certain types of trigonometric equations.

Example 1.—Solve the equation

$$\cos\theta+\cos 3\theta = 2\cos 2\theta.$$

This equation can be written

$$2\cos 2\theta\cos\theta = 2\cos 2\theta,$$

so that either $\cos 2\theta = 0$ or $\cos\theta = 1$.

Hence either $\quad 2\theta = (n+\tfrac{1}{2})\pi \quad$ or $\quad \theta = 2n\pi,$

where n is any integer.

Thus $$\theta = \tfrac{1}{2}(n+\tfrac{1}{2})\pi \quad \text{or} \quad 2n\pi.$$

Example 2.—Solve the equation

$$\sin\theta+\sin 3\theta+\sin 5\theta = 0.$$

Here $\quad L.H.S. = 2\sin 3\theta\cos 2\theta + \sin 3\theta,$

so that either $\sin 3\theta = 0$ or $\cos 2\theta = -\tfrac{1}{2}$.

Thus, $\quad 3\theta = n\pi \quad$ or $\quad 2\theta = 2n\pi \pm \tfrac{2}{3}\pi,$
and therefore $\quad \theta = \tfrac{1}{3}n\pi \quad$ or $\quad n\pi \pm \tfrac{1}{3}\pi.$

Example 3.—Solve the equation

$$\sin^2\lambda\theta - \sin^2(\lambda-1)\theta = \sin^2\theta.$$

$L.H.S. = \tfrac{1}{2}\{\cos 2(\lambda-1)\theta - \cos 2\lambda\theta\} = \sin(2\lambda-1)\theta\sin\theta,$
so that either $\sin\theta = 0$ or $\sin(2\lambda-1)\theta = \sin\theta.$

Thus $\quad \theta = n\pi$, or $(2\lambda-1)\theta = 2n\pi+\theta$ or $(2n+1)\pi-\theta,$
and therefore $\theta = n\pi$ or $n\pi/(\lambda-1)$ or $(2n+1)\pi/(2\lambda)$.

Example 4.—Solve the equation

$$\sin 4\theta - \sin 2\theta = \cos 3\theta.$$

Ans. $\quad \theta = 2n\pi+\tfrac{1}{6}\pi$ or $2n\pi+\tfrac{5}{6}\pi$ or $\tfrac{1}{3}(n+\tfrac{1}{2})\pi.$

§ 6. Series of Cosines and Sines of Angles in Arithmetical Progression

The series

$$\cos A + \cos(A + B) + \cos(A + 2B) + \ldots + \cos\{A + (n-1)B\},$$

$$\sin A + \sin(A + B) + \sin(A + 2B) + \ldots + \sin\{A + (n-1)B\}$$

can be summed by the *method of differences*. In applying this method each term of the series is expressed as the difference between two functions which are such that, on addition, all but two cancel. Thus, if the two sums above are denoted by C_n and S_n, respectively, it is found, on multiplying by $2\sin\frac{1}{2}B$, that

$$\begin{aligned}
2\sin\tfrac{1}{2}B \,.\, C_n &= \sin(A+\tfrac{1}{2}B)+\sin(A+\tfrac{3}{2}B)+\ldots+\sin\{A+(n-\tfrac{1}{2})B\}\\
&\quad -\sin(A-\tfrac{1}{2}B)-\sin(A+\tfrac{1}{2}B)-\ldots-\sin\{A+(n-\tfrac{3}{2})B\}\\
&= \sin\{A+(n-\tfrac{1}{2})B\}-\sin(A-\tfrac{1}{2}B)\\
&= 2\cos\left(A+\frac{n-1}{2}B\right)\sin\frac{nB}{2},
\end{aligned}$$

and

$$\begin{aligned}
2\sin\tfrac{1}{2}B \,.\, S_n &= \cos(A-\tfrac{1}{2}B)+\cos(A+\tfrac{1}{2}B)+\ldots+\cos\{A+(n-\tfrac{3}{2})B\}\\
&\quad -\cos(A+\tfrac{1}{2}B)-\cos(A+\tfrac{3}{2}B)-\ldots-\cos\{A+(n-\tfrac{1}{2})B\}\\
&= \cos(A-\tfrac{1}{2}B)-\cos\{A+(n-\tfrac{1}{2})B\}\\
&= 2\sin\left(A+\frac{n-1}{2}B\right)\sin\frac{nB}{2}.
\end{aligned}$$

Thus, on dividing by $2\sin\frac{1}{2}B$, we obtain the important formulæ

$$\cos A + \cos(A+B) + \cos(A+2B) + \ldots + \cos\{A+(n-1)B\} = \cos\left(A+\frac{n-1}{2}B\right)\frac{\sin\frac{1}{2}nB}{\sin\frac{1}{2}B}, \quad (9)$$

$$\sin A + \sin(A+B) + \sin(A+2B) + \ldots + \sin\{A+(n-1)B\} = \sin\left(A+\frac{n-1}{2}B\right)\frac{\sin\frac{1}{2}nB}{\sin\frac{1}{2}B}. \quad (10)$$

It should be noted that in these formulæ the first factors on the right are the cosine and the sine, respectively, of the arithmetic mean of the angles, or of half the sum of the first and the last; these are divided by the sine of half the common difference, and multiplied by the sine of n times half the common difference. Each formula can be obtained from the other by replacing A by A + 90°.

Example 1.—Show that

$$\cos\frac{\pi}{17} + \cos\frac{3\pi}{17} + \cos\frac{5\pi}{17} + \ldots + \cos\frac{15\pi}{17} = \tfrac{1}{2}.$$

Example 2.—Find the sum of the series

$$\cos^2\alpha + \cos^2(\alpha+\beta) + \cos^2(\alpha+2\beta) + \ldots + \cos^2\{\alpha+(n-1)\beta\}.$$

The series is equal to

$$\tfrac{1}{2}(1+\cos 2\alpha) + \tfrac{1}{2}\{1+\cos 2(\alpha+\beta)\} + \ldots + \tfrac{1}{2}[1+\cos 2\{\alpha+(n-1)\beta\}]$$

and therefore to

$$\tfrac{1}{2}n + \tfrac{1}{2}\cos\{2\alpha+(n-1)\beta\}\frac{\sin n\beta}{\sin\beta}.$$

Example 3.—Sum to n terms the series

$$\cos^3\alpha + \cos^3(\alpha+\beta) + \cos^3(\alpha+2\beta) + \ldots.$$

Put $\cos^3\alpha = \tfrac{1}{4}(\cos 3\alpha + 3\cos\alpha)$,

and similarly with the other terms. Thus, the series is equal to

$$\tfrac{1}{4}\{\cos 3\alpha + \cos 3(\alpha+\beta) + \cos 3(\alpha+2\beta) + \ldots + \cos 3[\alpha+(n-1)\beta]\}$$
$$+ \tfrac{3}{4}\{\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos[\alpha+(n-1)\beta]\}$$
$$= \tfrac{1}{4}\cos 3\left(\alpha+\frac{n-1}{2}\beta\right)\frac{\sin\frac{3}{2}n\beta}{\sin\frac{3}{2}\beta} + \tfrac{3}{4}\cos\left(\alpha+\frac{n-1}{2}\beta\right)\frac{\sin\frac{1}{2}n\beta}{\sin\frac{1}{2}\beta}.$$

Example 4.—The terms of the following set

$$\sin\alpha,\ \sin(\alpha+\beta),\ \sin(\alpha+2\beta),\ \ldots,\ \sin\{\alpha+(2n-1)\beta\}$$

are multiplied together in pairs, the first and the last, the second and the second last, and so on, and the products are added. Show that the sum of the series so obtained is

$$\tfrac{1}{4}\sin 2n\beta\,\mathrm{cosec}\,\beta - \tfrac{1}{2}n\cos\{2\alpha+(2n-1)\beta\}.$$

Example 5.—The points A, B, C, . . . are the vertices of a regular n-sided polygon inscribed in a circle of centre O and radius r. If perpendiculars PL, PM, PN . . . are drawn to

OA, OB, OC, . . . respectively from a point P on the circle show that

$$\text{(i) } LP^2 + MP^2 + NP^2 + \ldots = \tfrac{1}{2}nr^2,$$
$$\text{(ii) } AP^2 + BP^2 + CP^2 + \ldots = 2nr^2,$$
$$\text{(iii) } AB^2 + AC^2 + AD^2 + \ldots = 2nr^2.$$

[Let $\angle AOP = \theta$; then, if A, B, C, . . . lie round the circle in the counter-clockwise order,

$$\angle BOP = \theta - 2\pi/n, \quad \angle COP = \theta - 4\pi/n, \ \ldots,$$

so that $LP = r \sin \theta$, $MP = r \sin (\theta - 2\pi/n)$, ...]

Example 6.—A regular polygon of n sides is circumscribed to a circle of centre O and radius r, and from any point P within the circle perpendiculars are drawn to the sides. If OP is of length k, show that the sum of the squares on these perpendiculars is

$$n(r^2 + \tfrac{1}{2}k^2).$$

§ 7. Further Applications of the Method of Differences to the Summation of Series

If, for the terms of the series

$$u_1 + u_2 + \ldots + u_n,$$

a difference formula

$$u_r = f(r) - f(r-1)$$

can be found, the sum of the series is

$$\sum_{r=1}^{n} \{f(r) - f(r-1)\} = f(n) - f(0).$$

The method presents no difficulty if the difference formula is given, or if the sum to n terms is given. In the latter case, if the sum to n terms is $\phi(n)$, then

$$u_r = \phi(r) - \phi(r-1),$$

and this is the difference formula.

Example 1.—Show that

$$\operatorname{cosec} \alpha = \cot \tfrac{1}{2}\alpha - \cot \alpha,$$

and sum to n terms the series

$$\text{(i) } \operatorname{cosec} \alpha + \operatorname{cosec} 2\alpha + \operatorname{cosec} 4\alpha + \ldots,$$
$$\text{(ii) } \operatorname{cosec} \alpha + \operatorname{cosec} \tfrac{1}{2}\alpha + \operatorname{cosec} \tfrac{1}{4}\alpha + \ldots.$$

Ans. (i) $\cot \frac{1}{2}\alpha - \cot (2^{n-1}\alpha)$,
(ii) $\cot (\alpha/2^n) - \cot \alpha$.

Note.—The reader should check his results by putting $n = 1$.

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Put

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$$= \tfrac{1}{4}\cos 3\left(\alpha + \frac{n-1}{2}\beta\right)\frac{\sin\frac{3}{2}n\beta}{\sin\frac{3}{2}\beta} + \tfrac{3}{4}\cos\left(\alpha+\frac{n-1}{2}\beta\right)\frac{\sin\frac{1}{2}n\beta}{\sin\frac{1}{2}\beta}.$$

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Example 1.—Show that

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and sum to n terms the series

$$\text{(i) } \operatorname{cosec}\alpha + \operatorname{cosec}2\alpha + \operatorname{cosec}4\alpha + \dots,$$
$$\text{(ii) } \operatorname{cosec}\alpha + \operatorname{cosec}\tfrac{1}{2}\alpha + \operatorname{cosec}\tfrac{1}{4}\alpha + \dots.$$

Ans. (i) $\cot\frac{1}{2}\alpha - \cot(2^{n-1}\alpha)$,
(ii) $\cot(\alpha/2^n) - \cot\alpha$.

Note.—The reader should check his results by putting $n = 1$.

Example **2.**—Prove that

$$\tan \alpha = \cot \alpha - 2 \cot 2\alpha,$$

and find the sum to n terms of the series

$$\text{(i)} \quad \tan \alpha + 2 \tan (2\alpha) + 2^2 \tan (2^2\alpha) + \ldots,$$

$$\text{(ii)} \quad \tfrac{1}{2} \tan \frac{\alpha}{2} + \frac{1}{2^2} \tan \frac{\alpha}{2^2} + \frac{1}{2^3} \tan \frac{\alpha}{2^3} + \ldots.$$

Ans. (i) $\cot \alpha - 2^n \cot (2^n\alpha)$; (ii) $\dfrac{1}{2^n} \cot \left(\dfrac{\alpha}{2^n}\right) - \cot \alpha.$

Example **3.**—Sum to $2n$ terms the series

$$\tan \alpha + \cot \alpha + \tan 2\alpha + \cot 2\alpha + \tan 4\alpha + \cot 4\alpha + \ldots$$

Ans. $2 \cot \alpha - 2 \cot (2^n\alpha)$.

Example **4.**—Prove that

$$\sin \theta \sec 3\theta = \tfrac{1}{2}(\tan 3\theta - \tan \theta),$$

and sum to n terms the series

$$\text{(i)} \quad \sin \theta \sec 3\theta + \sin 3\theta \sec (3^2\theta) + \sin (3^2\theta) \sec (3^3\theta) + \ldots,$$

$$\text{(ii)} \quad \sin \theta \sec 3\theta + \sin \tfrac{1}{3}\theta \sec \theta + \sin\left(\frac{1}{3^2}\theta\right) \sec (\tfrac{1}{3}\theta) + \ldots.$$

Ans. (i) $\tfrac{1}{2}\{\tan (3^n\theta) - \tan \theta\}$; (ii) $\tfrac{1}{2}\left\{\tan 3\theta - \tan \left(\dfrac{1}{3^{n-1}}\theta\right)\right\}$.

The Sum to Infinity.—If S_n, the sum to n terms of a series, tends to a value S as n tends to infinity; that is, if, no matter how small we wish to make the difference between S_n and S, we can do so by taking n large enough, S is called the *sum to infinity* of the series. We write this

$$S_n \to S \quad \text{as} \quad n \to \infty,$$

where the arrowhead is the symbol for "tends to."

Example **5.**—Show that the sum to infinity of the series (ii) of *Example* 4 is $\tfrac{1}{2} \tan 3\theta$.

$$\left[\text{When } n \to \infty, \tan \left(\frac{1}{3^{n-1}}\theta\right) \to \tan 0 = 0.\right]$$

§ 8. Recurring Series

The series

$$\cos A + x \cos (A + B) + x^2 \cos (A + 2B) + \ldots + x^{n-1} \cos \{A + (n - 1)B\},$$

$$\sin A + x \sin (A + B) + x^2 \sin (A + 2B) + \ldots + x^{n-1} \sin \{A + (n - 1)B\}$$

are recurring series with $1 - 2x\cos B + x^2$ as their *scale of relation*. To find their sums, denoted by C_n and S_n respectively, multiply each series by the scale of relation. Thus

$$\begin{aligned}
&(1 - 2x\cos B + x^2)C_n \\
&= \cos A + x\cos(A+B) + x^2\cos(A+2B) + \ldots \\
&\qquad + x^r\cos(A+rB) + \ldots \\
&\qquad - 2x\cos A\cos B - 2x^2\cos(A+B)\cos B - \ldots \\
&\qquad - 2x^r\cos\{A+(r-1)B\}\cos B - \ldots \\
&\qquad + x^2\cos A + \ldots + x^r\cos\{A+(r-2)B\} + \ldots \\
&\ldots + x^{n-1}\cos\{A+(n-1)B\} \\
&\qquad - 2x^{n-1}\cos\{A+(n-2)B\}\cos B \\
&\qquad - 2x^n\cos\{A+(n-1)B\}\cos B \\
&\qquad + x^{n-1}\cos\{A+(n-3)B\} + x^n\cos\{A+(n-2)B\} \\
&\qquad + x^{n+1}\cos\{A+(n-1)B\} \\
&= \cos A - x\cos(A-B) - x^n\cos(A+nB) \\
&\qquad + x^{n+1}\cos\{A+(n-1)B\}.
\end{aligned}$$

Similarly

$$\begin{aligned}
(1 - 2x\cos B + x^2)S_n &= \sin A - x\sin(A-B) \\
&\quad - x^n\sin(A+nB) + x^{n+1}\sin\{A+(n-1)B\}.
\end{aligned}$$

Example.—Sum to n terms the series

(i) $p\cos A + (p+q)\cos(A+B) + (p+2q)\cos(A+2B) + \ldots$,
(ii) $p\sin A + (p+q)\sin(A+B) + (p+2q)\sin(A+2B) + \ldots$.

Denote these sums by C_n and S_n respectively: then

$$\begin{aligned}
2\cos B\,.\,C_n &= p\{\cos(A-B) + \cos(A+B)\} \\
&\quad + (p+q)\{\cos A + \cos(A+2B)\} + \ldots \\
&\quad + \{p+(n-1)q\}[\cos\{A+(n-2)B\} + \cos(A+nB)] \\
&= p\cos(A-B) + (p+q)\cos A \\
&\quad + 2(p+q)\cos(A+B) \\
&\quad + 2(p+2q)\cos(A+2B) + \ldots \\
&\ldots + 2\{p+(n-2)q\}\cos\{A+(n-2)B\} \\
&\quad + \{p+(n-2)q\}\cos\{A+(n-1)B\} \\
&\quad + \{p+(n-1)q\}\cos(A+nB),
\end{aligned}$$

and therefore

$$\begin{aligned}
2(1-\cos B)C_n &= -p\cos(A-B) + (p-q)\cos A \\
&\quad + (p+nq)\cos\{A+(n-1)B\} \\
&\quad - \{p+(n-1)q\}\cos(A+nB).
\end{aligned}$$

Similarly,

$$\begin{aligned}
2(1-\cos B)S_n &= -p\sin(A-B) + (p-q)\sin A \\
&\quad + (p+nq)\sin\{A+(n-1)B\} \\
&\quad - \{p+(n-1)q\}\sin(A+nB).
\end{aligned}$$

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EXAMPLES VII

1. Show that

(i) $4\sin 5A\cos 3A\cos 2A = \sin 4A + \sin 6A + \sin 10A$,
(ii) $\sin A(\sin A + \sin 3A + \sin 5A) = \sin^2 3A$,
(iii) $\cos 3A = \sin 3A + (\cos A + \sin A)(1 - 2\sin 2A)$,
(iv) $\tan A + \tan (A + 60°) + \tan (A - 60°) = 3\tan 3A$,
(v) $(\cot A - \cot 3A)(\sin^2 2A - \sin^2 A) = \sin 2A$.

2. If $\sin A + \sin B = p$, $\cos A + \cos B = q$, show that

$$\frac{1}{p}\sin\tfrac{1}{2}(A + B) = \frac{1}{q}\cos\tfrac{1}{2}(A + B) = \frac{2}{p^2 + q^2}\cos\tfrac{1}{2}(A - B).$$

3. If
$$\lambda = \frac{\cos\theta + \sin\phi}{\cos\phi + \sin\theta},$$
show that
$$\frac{1 - \lambda}{1 + \lambda} = \tan\tfrac{1}{2}(\theta - \phi).$$

4. If
$$\frac{\cos A}{\cos B} = \frac{\sin (C - \theta)}{\sin (C + \theta)},$$
show that $\tan\frac{1}{2}(A + B)\tan\frac{1}{2}(A - B) = \cot C\tan\theta$.

5. Establish the identities

(i) $\tan\frac{1}{2}(A + B) + \tan\frac{1}{2}(A - B) = \dfrac{2\sin A}{\cos A + \cos B}$,
(ii) $4\sin A\sin (A + \frac{1}{3}\pi)\sin (A + \frac{2}{3}\pi) = \sin 3A$,
(iii) $4\cos A\cos (\frac{2}{3}\pi - A)\cos (\frac{2}{3}\pi + A) = \cos 3A$,
(iv) $\cot (A + 15°) - \tan (A - 15°) = \dfrac{4\cos 2A}{2\sin 2A + 1}$,
(v) $\cos (B + C)\cos (B - C) - \cos (A + C)\cos (A - C)$
$= \sin (A + B)\sin (A - B)$,
(vi) $\sin A\cos 3A + \sin 2A\sin^2 A = \frac{1}{4}\sin 4A$,
(vii) $\cos 2A\cos 2B + \frac{1}{4}\sin 4A\sin 4B$
$= \cos^4 (A - B) - \sin^4 (A + B)$,
(viii) $\sin^3 A\sin 2A + \cos^3 A\cos 2A$
$= \frac{1}{2}\cos A + \frac{1}{2}\cos 2A\cos 3A$.

6. Prove that $x^2 - x\cos (A + B) + 1$ is a factor of
$$2x^4 + 4x^3\sin A\sin B - x^2(\cos 2A + \cos 2B) + 4x\cos A\cos B - 2,$$
and find the other factor.

Ans. $2x^2 + 2x\cos (A - B) - 2$.

7. Solve the quadratic equation in x
$$x^2(\cos 2A + \cos 2B) - 2x\sin 2A - \cos 2A + \cos 2B = 0.$$

Ans. $\tan (A + B)$, $\tan (A - B)$.

8. Establish the identities

(i) $\dfrac{\sec A - \sec B}{\tan A - \tan B} = \dfrac{\tan \frac{1}{2}A + \tan \frac{1}{2}B}{1 + \tan \frac{1}{2}A \tan \frac{1}{2}B}$,

(ii) $\dfrac{\sin A - \sin 3A}{\cos A + \cos 3A} + \dfrac{\sin A + \sin 3A}{\cos A - \cos 3A} = 2 \cot 2A$,

(iii) $\sin A(1 - \cos B) + \sin B(1 - \cos A)$
$= 4 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}(A + B)$,

(iv) $\dfrac{\sin^2 A - \sin^2 B}{\sin A \cos A - \sin B \cos B} = \tan (A + B)$,

(v) $\dfrac{2 (\sin 2A + \sin 2B)}{1 + \cos 2A + \cos 2B + \cos 2(A - B)} = \tan A + \tan B$.

9. Prove that

(i) $1 + \dfrac{\cos 2\theta + \cos 6\theta}{\cos 4\theta} = \dfrac{\sin 3\theta}{\sin \theta}$,

(ii) $\dfrac{\sin 3\theta - \sin \theta}{\sec 3\theta + \sec \theta} = \dfrac{\sin \theta}{\sec 3\theta}$,

(iii) $\dfrac{\sin \theta - 2 \sin 3\theta + \sin 5\theta}{\cos \theta - \cos 5\theta} = - \tan \theta$.

10. Without using tables, prove that

(i) $\sin 19° + \sin 41° + \sin 83°$
$= \sin 23° + \sin 37° + \sin 79°$,

(ii) $\sin 15° \cos 39° + \sin 31° \sin 83° + \sin 27° \cos 65°$
$= \cos 17° \cos 19°$,

(iii) $\sin 40° \sin 80° + \sin 80° \sin 160° + \sin 160° \sin 320°$
$= \frac{3}{4}$,

(iv) $\cos 40° \cos 80° \cos 160° = - \frac{1}{8}$,

(v) $\cos 14° \sin 62° - \cos 67° \sin 37° - \sin 9° \sin 51° = \frac{1}{2}$,

(vi) $\cos 77° \cos 43° + \sin 16° \sin 18° = \sin 31° \cos 61°$,

(vii) $\sin 79° \sin 92° - \cos 14° \cos 59° - \sin 27° \cos 72°$
$= \frac{1}{2} \sin 43°$,

(viii) $\sin 52° \sin 68° - \sin 47° \cos 77° - \cos 65° \cos 81° = \frac{1}{2}$,

(ix) $\cos 15° \cos 34° + \sin 42° \cos 151° - \sin 63° \sin 14° = 0$,

(x) $\sin 56° \cos 32° - \sin 48° \sin 46° - \sin 76° \cos 80° = 0$,

(xi) $\cos 250° \cos 72° = \sin 85° \sin 267° - \sin 75° \cos 203°$,

(xii) $\cos 28° = \cos 37° \sin 25° + \sin 41° \sin 13°$
$+ \cos 12° \cos 66°$.

11. Given that $\sqrt{6} = 2{\cdot}44949$ and $\sqrt{2} = 1{\cdot}41421$, prove without reference to tables that

$\cos 71° \sin 94° + \sin 40° \cos 93° - \cos 31° \cos 74°$
$+ \sin 15° \sin 52° = 0{\cdot}25882$.

12. Show that

(i) $\cos^2(A - B) = \cos^2 A - \sin^2 B + 2 \sin A \sin B \cos(A - B)$,

(ii) $\cos^2 A + \cos^2 B - 2 \cos A \cos B \cos(A + B) = \sin^2(A + B)$,

(iii) $\sin^2 A + \sin^2 B + \cos^2(A + B) + 2 \sin A \sin B \cos(A + B) = 1$,

(iv) $\cos(3A - B) - \cos(3B - A) = 4(\sin^2 B - \sin^2 A) \cos(A - B)$,

(v) $\sin^2 A \sin 2B + \cos^2 B \sin 2A = 2 \sin A \cos B \cos(A - B)$,

(vi) $8 \sin 3A \cos^3 A - 8 \sin 2A \cos^3 A = \sin 3A + \sin 5A$.

13. P is any point on the shorter arc AC of the circumcircle of an equilateral triangle ABC. Prove trigonometrically that the chord PB is equal to the sum of the chords PA and PC.

14. If $A + B + C = 180°$ show that

(i) $\sin 2A + \sin 2B - \sin 2C = 4 \cos A \cos B \sin C$,

(ii) $\cos 2A + \cos 2B + \cos 2C + 1 + 4 \cos A \cos B \cos C = 0$,

(iii) $\cos 2A + \cos 2B - \cos 2C + 4 \sin A \sin B \cos C = 1$,

(iv) $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C$,

(v) $\sin A + \sin B + \sin C = 4 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C$,

(vi) $\sin A + \sin B - \sin C = 4 \sin \frac{1}{2}A \sin \frac{1}{2}B \cos \frac{1}{2}C$,

(vii) $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C$,

(viii) $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C$,

(ix) $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2 \sin A \sin B \cos C$,

(x) $\cos^2 \frac{1}{2}A + \cos^2 \frac{1}{2}B - \cos^2 \frac{1}{2}C = 2 \cos \frac{1}{2}A \cos \frac{1}{2}B \sin \frac{1}{2}C$,

(xi) $\cos^2 \frac{1}{2}A + \cos^2 \frac{1}{2}B + \cos^2 \frac{1}{2}C = 2 + 2 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C$,

(xii) $\sin^2(A + 45°) + \sin^2(B + 45°) - \cos^2 C = 2 \sin C \sin(A + 45°) \sin(B + 45°)$,

(xiii) $\cos^2 2A + \cos^2 2B + \sin^2 2C = 2 + 2 \sin 2A \sin 2B \cos 2C$,

(xiv) $\sin 2B \cos 2C + \sin 2C \cos 2A + \sin 2A \cos 2B = -2 \sin A \sin B \sin C - 2 \sin(B - C) \sin(C - A) \sin(A - B)$,

(xv) $\sin(120° - A) + \sin(120° - B) + \sin(120° - C) = 4 \cos(60° - \frac{1}{2}A) \cos(60° - \frac{1}{2}B) \cos(60° - \frac{1}{2}C)$,

(xvi) $\sin(2A - B) + \sin(2B - C) + \sin(2C - A) = 4 \cos(A - \frac{1}{2}B) \cos(B - \frac{1}{2}C) \cos(C - \frac{1}{2}A)$,

(xvii) $\sin 3A + \sin 3B + \sin 3C = -4 \cos \frac{3}{2}A \cos \frac{3}{2}B \cos \frac{3}{2}C$,

(xviii) $\sin^3 A + \sin^3 B + \sin^3 C = 3 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C + \cos \frac{3}{2}A \cos \frac{3}{2}B \cos \frac{3}{2}C$,

(xix) $\dfrac{\sin 2A + \sin 2B - \sin 2C}{\sin 2A - \sin 2B + \sin 2C} = \dfrac{\tan C}{\tan B}$,

(xx) $\dfrac{\sin A + \sin B - \sin C}{\sin A + \sin B + \sin C} = \tan \frac{1}{2}A \tan \frac{1}{2}B$,

(xxi) $\sin 6A + \sin 6B + \sin 6C = 4 \sin 3A \sin 3B \sin 3C$.

15. Establish the following identities:

(i) $\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma)$
$= 4\cos\frac{1}{2}(\beta+\gamma)\cos\frac{1}{2}(\gamma+\alpha)\cos\frac{1}{2}(\alpha+\beta)$,

hence deducing that, in any triangle ABC,

(a) $\cos\frac{1}{2}A + \cos\frac{1}{2}B + \cos\frac{1}{2}C$
$= 4\sin\frac{1}{4}(\pi+A)\sin\frac{1}{4}(\pi+B)\sin\frac{1}{4}(\pi+C)$,

(b) $(\cos\frac{1}{2}A + \cos\frac{1}{2}B + \cos\frac{1}{2}C)^2$
$= 2(1+\sin\frac{1}{2}A)(1+\sin\frac{1}{2}B)(1+\sin\frac{1}{2}C)$,

(ii) $\sin 2(\alpha+\beta) + \sin 2(\alpha+\gamma) + \sin 2(\beta-\gamma)$
$= 4\sin(\alpha+\beta)\cos(\alpha+\gamma)\cos(\beta-\gamma)$,

(iii) $\cos^2\sigma + \cos^2(\sigma-\alpha) + \cos^2(\sigma-\beta) + \cos^2(\sigma-\gamma)$
$= 2 + 2\cos\alpha\cos\beta\cos\gamma$,

where $2\sigma = \alpha+\beta+\gamma$,

(iv) $\sin(\beta+\gamma-\alpha) + \sin(\gamma+\alpha-\beta)$
$+ \sin(\alpha+\beta-\gamma) - \sin(\alpha+\beta+\gamma)$
$= 4\sin\alpha\sin\beta\sin\gamma$,

(v) $\sin^2(x+\alpha) + \sin^2(x+\beta)$
$- 2\cos(\alpha-\beta)\sin(x+\alpha)\sin(x+\beta)$
$= \sin^2(\alpha-\beta)$,

(vi) $\cos(\beta+\gamma-\alpha) + \cos(\gamma+\alpha-\beta)$
$+ \cos(\alpha+\beta-\gamma) + \cos(\alpha+\beta+\gamma)$
$= 4\cos\alpha\cos\beta\cos\gamma$,

(vii) $\sin(\beta+\gamma-\alpha) + \sin(\gamma+\alpha-\beta)$
$- \sin(\alpha+\beta-\gamma) + \sin(\alpha+\beta+\gamma)$
$= 4\cos\alpha\cos\beta\sin\gamma$,

(viii) $\cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta\cos(\alpha+\beta)$
$= \sin^2(\alpha+\beta)$,

(ix) $\sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta\cos(\alpha+\beta)$
$= \sin^2(\alpha+\beta)$,

(x) $\sin(\beta-\gamma) + \sin(\gamma-\alpha) - \sin(\alpha-\beta)$
$= -4\sin\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\beta-\gamma)\cos\frac{1}{2}(\gamma-\alpha)$.

16. Show that, if

$$x\sin\alpha + y\sin\beta + z\sin\gamma = 0,$$

and

$$x\cos\alpha + y\cos\beta + z\cos\gamma = 0,$$

then

$$\frac{x}{\sin(\beta-\gamma)} = \frac{y}{\sin(\gamma-\alpha)} = \frac{z}{\sin(\alpha-\beta)},$$

and that each of these ratios is equal to

$$2xyz/\sqrt{\{(x+y+z)(x+y-z)(x-y+z)(-x+y+z)\}}.$$

17. If $\alpha+\beta+\gamma = \frac{1}{2}\pi$, prove that

$$\cos(\beta+\gamma-2\alpha)\sin(\alpha+2\beta) + \cos(\gamma+\alpha-2\beta)\sin(\beta+2\gamma)$$
$$+ \cos(\alpha+\beta-2\gamma)\sin(\gamma+2\alpha)$$
$$= 4\sin(\alpha+2\beta)\sin(\beta+2\gamma)\sin(\gamma+2\alpha).$$

18. Show that, if

$$\sec(\alpha+\beta+\gamma)+\sec(\beta+\gamma-\alpha)+\sec(\gamma+\alpha-\beta)$$
$$+\sec(\alpha+\beta-\gamma)=0,$$

either $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=2$, or else one of the angles α, β, γ is an odd multiple of a right angle.

19. Find all the angles between 0° and 360° that satisfy the following equations:

(i) $\cos 5\theta-\cos 3\theta+\sin 4\theta=0$.
Ans. 30°, 150°, $n \cdot 45°$ ($n=0, 1, 2, \ldots, 8$).

(ii) $\sin\theta+\sin 3\theta=\sin 4\theta$.
Ans. $n \cdot 90°$ ($n=0, 1, 2, 3, 4$), $n \cdot 120°$ ($n=0, 1, 2, 3$), 0°, 360°.

(iii) $\cos\theta+\sin 3\theta=\cos 2\theta$.
Ans. 90°, 135°, 315°, $n \cdot 120°$ ($n=0, 1, 2, 3$).

(iv) $\cos\theta+\cos 2\theta-\cos 4\theta-\cos 5\theta=0$.
Ans. $n \cdot 60°$ ($n=0, 1, 2, \ldots, 6$), 0°, 180°, 360°, 120°, 240°.

(v) $\sin 5\theta-2\sin 3\theta+3\sin\theta=0$.
Ans. $n \cdot 45°$ ($n=1, 3, 5, 7$), 0°, 180°, 360°, $n \cdot 30°$ ($n=1, 5, 7, 11$).

(vi) $\cos 5\theta+\cos 3\theta=\sin 3\theta+\sin\theta$.
Ans. 90°, 270°, 135°, 315°, $n \cdot 15°$ ($n=1, 5, 13, 17$).

(vii) $2\sin\theta-3\sin 3\theta+2\sin 5\theta=0$.
Ans. $n \cdot 60°$ ($n=0, 1, 2, \ldots, 6$), 20° 42′, 159° 18′, 200° 42′, 339° 18′.

(viii) $\sin\theta-\sin 2\theta+\sin 3\theta-\sin 4\theta=0$.
Ans. 90°, 270°, 0°, 360°, $n \cdot 36°$, ($n=1, 3, 5, 7, 9$).

(ix) $\cos 3\theta+\sin 4\theta+\cos\theta=0$.
Ans. $n \cdot 45°$ ($n=1, 3, 5, 7$), 90°, 270°, 210°, 330°.

(x) $\sin 3\theta-\sin\theta=\sin^2\theta-\cos^2\theta$.
Ans. $n \cdot 45°$ ($n=1, 3, 5, 7$), 210°, 330°.

(xi) $\cos 4\theta+\cos 3\theta+\cos 2\theta=0$.
Ans. $n \cdot 30°$ ($n=1, 3, 5, 7, 9, 11$), 120°, 240°.

(xii) $\sin 3\theta+\sin\theta-\sin 4\theta=0$.
Ans. $n \cdot 90°$ ($n=0, 1, 2, 3, 4$), 0, 360°, $n \cdot 120°$ ($n=0, 1, 2, 3$).

(xiii) $\sin\theta+\cos 3\theta=\cos 2\theta-\sin 4\theta$.
Ans. $n \cdot 72°$ ($n=0, 1, 2, 3, 4, 5$), 45°, 225°, 270°.

(xiv) $\sin\theta-2\sin 3\theta+\sin 5\theta=0$.
Ans. $n \cdot 60°$ ($n=0, 1, 2, \ldots, 6$), 0°, 180°, 360°.

(xv) $\cos\theta-\cos 2\theta+\cos 3\theta-\cos 4\theta=0$.
Ans. 90°, 270°, 0°, 360°, $n \cdot 72°$ ($n=0, 1, 2, \ldots, 5$).

(xvi) $\cos\theta+\sin\theta=4\cos\theta\sin^2\theta$.
Ans. 45°, 225°, $67\frac{1}{2}°+n \cdot 90°$ ($n=0, 1, 2, 3$).

20. Find expressions for all angles that satisfy the equation

$$\sin 2\theta + \sin 3\theta = \sin \theta.$$

Ans. $\theta = n\pi, (2n + 1)\pi, \frac{1}{3}(2n + 1)\pi$, where n is any integer.

21. Find the values of the constants a, b and c if

$$16 \sin^4 x \cos x = a \cos x + b \cos 3x + c \cos 5x.$$

Ans. $a = 2, b = -3, c = 1.$

22. Express $32 \cos^4 x \sin^2 x$ in the form

$$a + b \cos 2x + c \cos 4x + d \cos 6x.$$

Ans. $a = 2, b = 1, c = -2, d = -1.$

23. Show that

$$\tan n\theta = \frac{\sin \theta + \sin 3\theta + \sin 5\theta + \ldots \text{ to } n \text{ terms}}{\cos \theta + \cos 3\theta + \cos 5\theta + \ldots \text{ to } n \text{ terms}}.$$

24. Show that

$$\sin^2 x + \sin^2 2x + \sin^2 3x + \ldots + \sin^2 nx = \frac{2n + 1}{4} - \frac{\sin (2n + 1)x}{4 \sin x}.$$

25. Show that

(i) $\sin \frac{\pi}{15} + \sin \frac{5\pi}{15} + \sin \frac{9\pi}{15} + \ldots$ to 9 terms $= \sin \frac{\pi}{5}$,

(ii) $\cos^2 \frac{\pi}{17} + \cos^2 \frac{2\pi}{17} + \cos^2 \frac{3\pi}{17} + \ldots + \cos^2 \frac{8\pi}{17} = \frac{15}{4}.$

26. Show that, if $\alpha = 2\pi/(n + 1)$,

(i) $\sin \alpha + \sin 2\alpha + \sin 3\alpha + \ldots + \sin n\alpha = 0,$

(ii) $\sin^2 \alpha + \sin^2 2\alpha + \sin^2 3\alpha + \ldots + \sin^2 n\alpha = \frac{1}{2}(n + 1).$

27. Sum to n terms the series

$$\sin^2 \theta + \sin^2 3\theta + \sin^2 5\theta + \sin^2 7\theta + \ldots.$$

Ans. $\frac{1}{2}n - \frac{1}{4} \sin 4n\theta \operatorname{cosec} 2\theta.$

28. If $\alpha = 2\pi/n$, show that

$$1 + \cos \alpha + \cos 2\alpha + \cos 3\alpha + \ldots + \cos (n - 1)\alpha = 0.$$

29. Show that

$$8 \sin^4 \theta = \cos 4\theta - 4 \cos 2\theta + 3,$$

and deduce that, if $\alpha = \pi/40$,

$$\sin^4 \alpha + \sin^4 2\alpha + \sin^4 3\alpha + \ldots \text{ to 20 terms} = 8.$$

30. Sum to n terms the series

$$\cos A \sin 3A + \cos 2A \sin 6A + \cos 3A \sin 9A + \ldots.$$

Ans. $\frac{1}{2} \sin (2n + 2)A \sin 2nA \operatorname{cosec} 2A + \frac{1}{2} \sin (n + 1)A \sin nA \operatorname{cosec} A.$

31. Sum to n terms the series

$$\sin \alpha \sin 2\alpha + \sin 2\alpha \sin 3\alpha + \sin 3\alpha \sin 4\alpha + \ldots.$$

Ans. $\frac{1}{2} n \cos \alpha - \frac{1}{2} \cos (n + 2)\alpha \sin n\alpha \operatorname{cosec} \alpha.$

32. Show that, if $n > 2$,

$$\cos^2 \frac{\pi}{n} + \cos^2 \frac{3\pi}{n} + \cos^2 \frac{5\pi}{n} + \ldots + \cos^2 \frac{(2n-1)\pi}{n} = \frac{n}{2}.$$

33. Find the sum to n terms of the series

$$\sin\theta - \sin 2\theta + \sin 3\theta - \sin 4\theta + \ldots.$$

Ans. $-\sin\{\frac{1}{2}(n+1)(\theta+\pi)\}\sin\frac{1}{2}n(\theta+\pi)\operatorname{cosec}\frac{1}{2}(\theta+\pi)$.

34. Find the sum to n terms of the series

$$\cos A \sin 2A + \cos 2A \sin 3A + \cos 3A \sin 4A + \ldots.$$

Ans. $\frac{1}{2}n\sin A + \frac{1}{2}\sin(n+2)A\sin nA\operatorname{cosec} A$.

35. If n is a positive integer greater than 2, prove that the sum to n terms of the series

$$\sin^2\theta + \sin^2\left(\theta + \frac{2\pi}{n}\right) + \sin^2\left(\theta + \frac{4\pi}{n}\right) + \ldots$$

is $\frac{1}{2}n$.

36. Show that

$$\sin^2\frac{\pi}{2n} + \sin^2\frac{2\pi}{2n} + \sin^2\frac{3\pi}{2n} + \ldots \text{ to } n \text{ terms} = \tfrac{1}{2}(n+1).$$

37. Show that the sum of $(n-2)$ terms of the series whose rth term is

$$\sin\frac{r\pi}{2n}\cos\frac{(r+1)\pi}{2n}$$

is
$$\left(1 - n\sin^2\frac{\pi}{2n}\right)\Big/\left(2\sin\frac{\pi}{2n}\right).$$

38. Sum to n terms the series

$$\sin^3 x + \sin^3 2x + \sin^3 3x + \ldots.$$

Ans. $\frac{3}{4}\sin\frac{1}{2}(n+1)x\sin(\frac{1}{2}nx)\operatorname{cosec}\frac{1}{2}x$
$-\frac{1}{4}\sin\frac{3}{2}(n+1)x\sin(\frac{3}{2}nx)\operatorname{cosec}\frac{3}{2}x$.

39. Show that

$$2\sin\alpha - \sin 2\alpha = 4\sin\alpha\sin^2\tfrac{1}{2}\alpha,$$

and sum to n terms the series

$$\sin^2 x\sin 2x + \sin^2 2x\sin 4x + \sin^2 3x\sin 6x + \ldots$$

Ans. $\frac{1}{2}\sin(n+1)x\sin nx\operatorname{cosec} x$
$-\frac{1}{4}\sin(2n+2)x\sin 2nx\operatorname{cosec} 2x$.

40. $A_1, A_2, A_3, \ldots, A_n$ are the vertices, taken in order, of a regular polygon inscribed in a circle of radius r. Prove that the sum of the projections of $A_1A_3, A_1A_4, \ldots, A_1A_n$ on A_1A_2 is $r(n-2)\sin(\pi/n)$. Show also that

$$A_1A_2 + A_1A_3 + A_1A_4 + \ldots + A_1A_n = 2r\cot(\tfrac{1}{2}\pi/n).$$

41. The circumference of a semi-circle on the diameter AB is divided into n equal parts by the points $P_1, P_2, \ldots, P_{n-1}$. Show that the sum of the projections of the chords

$$AP_1, AP_2, \ldots, AP_{n-1}$$

on the diameter AB is equal to $\frac{1}{2}(n-1)AB$.

42. Prove that

$$\frac{1}{2\sin x}(\operatorname{cosec} 2x - \operatorname{cosec} 4x) = \frac{\cos 3x}{\sin 2x \sin 4x};$$

and sum to n terms the series

$$\frac{\cos 3x}{\sin 2x \sin 4x} + \frac{\cos 5x}{\sin 4x \sin 6x} + \frac{\cos 7x}{\sin 6x \sin 8x} + \ldots$$

Ans. $\{\operatorname{cosec} 2x - \operatorname{cosec}(2n+2)x\}/(2\sin x)$.

43. Show that

$$\sin^4\theta = \sin^2\theta - \tfrac{1}{4}\sin^2 2\theta;$$

and deduce the sum to n terms and to infinity of the series

$$\sin^4\theta + \tfrac{1}{4}\sin^4 2\theta + \frac{1}{4^2}\sin^4 4\theta + \frac{1}{4^3}\sin^4 8\theta + \ldots.$$

Ans. $\sin^2\theta - \dfrac{1}{4^n}\sin^2(2^n\theta)$, $\sin^2\theta$.

44. Show that

$$\tan(A+B) - \tan A = \sin B \sec A \sec(A+B);$$

and find the sum of the series

$$\sec A \sec 2A + \sec 2A \sec 3A + \ldots + \sec nA \sec(n+1)A.$$

Ans. $\operatorname{cosec} A\{\tan(n+1)A - \tan A\}$.

45. Prove that

(i) $\tan x \sec 2x + \tan 2x \sec 4x + \ldots$
$\qquad + \tan(2^{n-1}x)\sec(2^n x) = \tan(2^n x) - \tan x$,

(ii) $\tan x \tan 2x + \tan 2x \tan 3x + \ldots$
$\qquad + \tan nx \tan(n+1)x = \cot x \tan(n+1)x - (n+1)$.

46. Show that

$$\sec^2\theta \tan 2\theta = 2\tan 2\theta - 2\tan\theta;$$

and sum the series

$$\sec^2\theta\tan 2\theta + \sec^2\tfrac{1}{2}\theta\tan\theta + \sec^2\tfrac{1}{4}\theta\tan\tfrac{1}{2}\theta + \ldots$$

to n terms and to infinity.

Ans. $2\tan 2\theta - 2\tan(\theta/2^{n-1})$, $2\tan 2\theta$.

47. Sum the series

$$\sec\theta\tan\frac{\theta}{2} + \sec\frac{\theta}{2}\tan\frac{\theta}{2^2} + \sec\frac{\theta}{2^2}\tan\frac{\theta}{2^3} + \ldots$$

to n terms, and to infinity.

Ans. $\tan\theta - \tan(\theta/2^n)$, $\tan\theta$.

CHAPTER VIII

THE STANDARD LINEAR EQUATION

§ 1. Maxima and Minima of Linear Functions of Sines and Cosines

THE linear function

$$a\cos\theta + b\sin\theta + c \qquad . \quad . \quad (1)$$

can be put in the form

$$R\cos(\theta - \alpha) + c,$$

where R is positive, and α is called a *subsidiary angle*. To prove this we expand the cosine and compare the resulting expression

$$R\cos\theta\cos\alpha + R\sin\theta\sin\alpha + c$$

with (1). It is identically equal to (1) if

$$a = R\cos\alpha, \quad b = R\sin\alpha,$$

so that

$$a^2 + b^2 = R^2,$$

and therefore, since R is positive,

$$R = +\sqrt{(a^2 + b^2)}.$$

It follows that

$$\cos\alpha = \frac{a}{\sqrt{(a^2+b^2)}}, \sin\alpha = \frac{b}{\sqrt{(a^2+b^2)}}, \tan\alpha = \frac{b}{a}.$$

The value of α can be found from the table of tangents, the quadrant in which it lies being determined by the signs of $\cos\alpha$ and $\sin\alpha$.

The function (1) is therefore equivalent to

$$\sqrt{(a^2+b^2)}\cos(\theta - \alpha) + c \qquad . \quad . \quad . \quad (2)$$

and thus has its maximum and its minimum values when $\cos(\theta - \alpha)$ has the values $+1$ and -1, respectively. The maximum value is

$$\sqrt{(a^2+b^2)} + c,$$

corresponding to $\theta = \alpha + 2n\pi$;

and the minimum value is

$$-\sqrt{(a^2 + b^2)} + c,$$

corresponding to $\theta = \alpha + (2n + 1)\pi$.

Example 1.—Find the maximum and the minimum values of $6\cos\theta - 3\sin\theta + 7$, and the corresponding values of θ.

Here $\cos\alpha = \frac{2}{\sqrt{5}}$, $\sin\alpha = -\frac{1}{\sqrt{5}}$, $\tan\alpha = -\frac{1}{2}$,

so that α, being in the fourth quadrant, has the value $-26° 34'$. Thus the function is equal to

$$3\sqrt{5} \cdot \cos(\theta + 26° 34') + 7.$$

Its maximum value is $3\sqrt{5} + 7 = 13{\cdot}7082$, which occurs when $\theta = -26° 34' + n \cdot 360°$; and its minimum value is $-3\sqrt{5} + 7 = 0{\cdot}2918$, which occurs when $\theta = 153° 26' + n \cdot 360°$.

Graph of the Function.—The graph of the function (1), as is evident from the form (2), is of the same shape as the graph of the cosine. It may be drawn by tabulating the function (1) from sine and cosine tables for a suitable set of values of θ.

If, however, the values of θ at the turning-points have already been found, it is usually easy to obtain a fairly good graph without further use of the tables. Those values of θ are taken for which $\sin\theta$ or $\cos\theta$ vanishes, and form (1) is employed to calculate y; in addition those values of θ are taken for which $\cos(\theta - \alpha)$ has the values $0, \pm 1, \pm \frac{1}{2}$, and form (2) is then employed to calculate y. Such values of θ as $\pm 30°$, $\pm 60°$, $\pm 45°$, may also be used.

Example 2.—Draw the graph of

$$y = 6\cos\theta - 3\sin\theta + 7 = 3\sqrt{5} \cdot \cos(\theta + 26° 34') + 7$$

from $-180°$ to $+180°$.

(This is the function considered in *Example* 1. Note that $3\sqrt{5} = 6{\cdot}7082$).

θ	$-180°$	$-146° 34'$	$-116° 34'$	$-90°$	$-26° 34'$	0	$33° 26'$	$63° 26'$	$90°$	$153° 26'$	$180°$
y	1	3·65	7	10	13·71	13	10·35	7	4	·29	1

From the symmetry of the curve about the lines $\theta = -26° 34'$ and $\theta = 153° 26'$ it follows that further points on the curve can be found by dropping perpendiculars on these lines from points already found on the curve and producing each of these perpendiculars its own length. Thus, for instance, in Fig. 1

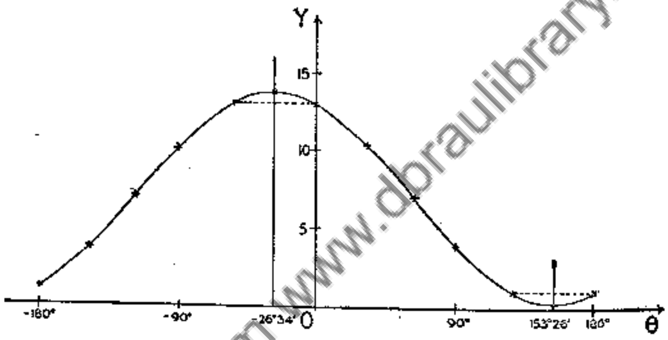

FIG. 1.

useful points on the graph are obtained by finding the image of the point (0, 13) in the line $\theta = -26° 34'$, and the image of the point (180°, 1) in the line $\theta = 153° 26'$.

§ 2. Solution of Linear Equations by the Subsidiary Angle Method

Two methods of solving the standard linear equation

$$a \cos \theta + b \sin \theta = c \quad . \quad . \quad . \quad (3)$$

will now be given. In the first of these use is made of the subsidiary angle. It may be assumed that none of the constants a, b, c is zero. It may also be assumed that c is positive: if c is negative it can be made positive by changing the signs on both sides of the equation.

If equation (3) is divided by $\sqrt{(a^2 + b^2)}$, it becomes

$$\cos(\theta - \alpha) = \frac{c}{\sqrt{(a^2 + b^2)}},$$

where α is determined as in § 1.

In order that this equation should have real roots the condition

$$c \leqq \sqrt{(a^2 + b^2)},$$

or

$$c^2 \leqq a^2 + b^2 \quad . \quad . \quad . \quad . \quad (4)$$

must be satisfied. An angle β can then be obtained from the cosine table to satisfy the equation

$$\cos\beta = \frac{c}{\sqrt{(a^2 + b^2)}} \quad . \quad . \quad . \quad (5)$$

Then $\qquad \cos(\theta - \alpha) = \cos\beta;$

so that $\qquad \theta - \alpha = \pm\beta + 2n\pi,$

or $\qquad \theta = \alpha \pm \beta + 2n\pi. \quad . \quad . \quad . \quad (6)$

The roots which lie in the range from 0 to 2π are called the fundamental roots of the equation.

Note.—In solving the equation it is not *necessary* to make c positive; but it is advantageous to do so, as β can then be taken positive and acute, with the result that its value can be obtained straight from the tables.

Example I.—Solve the equation

$$3\cos\theta + 5\sin\theta = 4.$$

Here $\cos\alpha = \dfrac{3}{\sqrt{(34)}}$, $\sin\alpha = \dfrac{5}{\sqrt{(34)}}$, $\tan\alpha = \frac{5}{3} = 1{\cdot}66667$,

so that $\alpha = 59^\circ\ 2'$. Hence

$$\cos(\theta - 59^\circ\ 2') = \frac{4}{\sqrt{(34)}},$$

and therefore

$$\begin{aligned} \log\cos(\theta - 59^\circ\ 2') &= \log 4 - \tfrac{1}{2}\log(34) \\ &= 0{\cdot}60206 - \tfrac{1}{2} \times 1{\cdot}53148 \\ &= 0{\cdot}60206 - 0{\cdot}76574 \\ &= \bar{1}{\cdot}83632 \\ &= \log\cos(46^\circ\ 41'). \end{aligned}$$

Thus

$$\begin{aligned} \theta &= 59^\circ\ 2' \pm 46^\circ\ 41' + n\,.\,360^\circ \\ &= 105^\circ\ 43' + n\,.\,360^\circ \text{ or } 12^\circ\ 21' + n\,.\,360^\circ. \end{aligned}$$

Example 2.—Solve the equation

$$8 \cos \theta - 5 \sin \theta + 9 = 0.$$

This may be written

$$-\frac{8}{\sqrt{(89)}} \cos \theta + \frac{5}{\sqrt{(89)}} \sin \theta = \frac{9}{\sqrt{(89)}}.$$

Thus

$$\cos \alpha = -\frac{8}{\sqrt{(89)}}, \quad \sin \alpha = \frac{5}{\sqrt{(89)}}, \quad \tan \alpha = -\tfrac{5}{8} = -0{\cdot}62500,$$

so that, since α is in the second quadrant,

$$\alpha = 180^\circ - 32^\circ = 148^\circ.$$

Hence

$$\cos (\theta - 148^\circ) = \frac{9}{\sqrt{(89)}},$$

and therefore

$$\begin{aligned} \log \cos (\theta - 148^\circ) &= \log 9 - \tfrac{1}{2} \log (89) \\ &= 0{\cdot}95424 - \tfrac{1}{2} \times 1{\cdot}94939 \\ &= 0{\cdot}95424 - 0{\cdot}97470 \\ &= \bar{1}{\cdot}97954 = \log \cos (17^\circ\ 27'). \end{aligned}$$

Thus

$$\begin{aligned} \theta &= 148^\circ \pm 17^\circ\ 27' + n\,.\,360^\circ \\ &= 165^\circ\ 27' + n\,.\,360^\circ \text{ or } 130^\circ\ 33' + n\,.\,360^\circ. \end{aligned}$$

Alternative Method.—If the reader prefers to use sines rather than cosines, the subsidiary angle α may be chosen to satisfy the equations

$$\sin \alpha = \frac{a}{\sqrt{(a^2 + b^2)}}, \quad \cos \alpha = \frac{b}{\sqrt{(a^2 + b^2)}}, \quad \tan \alpha = \frac{a}{b}.$$

Then (3) gives $\sin (\theta + \alpha) = \dfrac{c}{\sqrt{(a^2 + b^2)}}$,

so that (c being positive), if β is the positive acute angle for which

$$\sin \beta = \frac{c}{\sqrt{(a^2 + b^2)}},$$

$$\theta + \alpha = \beta + 2n\pi \quad \text{or} \quad (\pi - \beta) + 2n\pi$$

and therefore

$$\theta = -\alpha + \beta + 2n\pi \quad \text{or} \quad -\alpha + (\pi - \beta) + 2n\pi.$$

Example 3.—Solve the equation

$$7 \sin \theta - 4 \cos \theta + 3 = 0.$$

This may be written

$$4\cos\theta - 7\sin\theta = 3,$$

so that

$$\sin\alpha = \frac{4}{\sqrt{(65)}}, \quad \cos\alpha = -\frac{7}{\sqrt{(65)}}, \quad \tan\alpha = -\tfrac{4}{7} = -0{\cdot}57143.$$

Thus $$\alpha = 180^\circ - 29^\circ\ 45' = 150^\circ\ 15',$$

and $$\sin(\theta + 150^\circ\ 15') = \frac{3}{\sqrt{(65)}}.$$

Hence

$$\begin{aligned}\log\sin(\theta + 150^\circ\ 15') &= \log 3 - \tfrac{1}{2}\log(65)\\ &= 0{\cdot}47712 - \tfrac{1}{2}\times 1{\cdot}81291\\ &= 0{\cdot}47712 - 0{\cdot}90645\\ &= \bar{1}{\cdot}57067 = \log\sin(21^\circ\ 51').\end{aligned}$$

Therefore

$$\begin{aligned}\theta &= -150^\circ\ 15' + 21^\circ\ 51' + n\,.\,360^\circ\\ &\quad \text{or } -150^\circ\ 15' + 158^\circ\ 9' + n\,.\,360^\circ\\ &= -128^\circ\ 24' + n\,.\,360^\circ \text{ or } 7^\circ\ 54' + n\,.\,360^\circ.\end{aligned}$$

§ 3. Solution of the Equation as a Quadratic in $\tan\frac{1}{2}\theta$.

Formulæ (14) and (15) of Chapter VI may be written

$$\cos\theta = \frac{1-t^2}{1+t^2}, \quad \sin\theta = \frac{2t}{1+t^2} \qquad . \qquad . \qquad (7)$$

where $t = \tan\frac{1}{2}\theta$. If these values for $\cos\theta$ and $\sin\theta$ are substituted in equation (3), it becomes

$$a(1-t^2) + b\,.\,2t = c(1+t^2),$$

or $$(a+c)t^2 - 2bt - (a-c) = 0 \qquad . \qquad . \qquad (8)$$

This equation has real roots if

$$4b^2 + 4(a+c)(a-c) \geqq 0,$$

or if $$c^2 \leqq a^2 + b^2,$$

which is the condition obtained in § 2.

On solving (8) two roots t_1 and t_2 are obtained. If values of λ and μ for which

$$\tan\lambda = t_1, \quad \tan\mu = t_2$$

are then obtained from the tangent tables,

$$\tfrac{1}{2}\theta = \lambda + n\pi \quad \text{or} \quad \mu + n\pi,$$

so that $$\theta = 2\lambda + 2n\pi \quad \text{or} \quad 2\mu + 2n\pi.$$

Example.—Solve the equation

$$4 \cos \theta + 2 \sin \theta + 3 = 0.$$

By means of (7) this may be written

$$4(1 - t^2) + 2 \,.\, 2t + 3(1 + t^2) = 0,$$

or

$$t^2 - 4t - 7 = 0.$$

On solving it is found that

$$t = 2 \pm \sqrt{(11)} = 2 \pm 3{\cdot}31663 = 5{\cdot}31663 \text{ or } -1{\cdot}31663.$$

Hence $\tfrac{1}{2}\theta = 79° \, 21' + n \,.\, 180°$ or $- 52° \, 47' + n \,.\, 180°$,

and $\theta = 158° \, 42' + n \,.\, 360°$ or $- 105° \, 34' + n \,.\, 360°$.

The reader might find it helpful to solve this example by the method of § 2, and to solve by this method the three examples worked out in § 2.

§ 4. Geometrical Verification

The solutions of a linear equation can be verified geometrically by the following method, which is due to Dr. John McWhan.

In the equation (3) make c positive by, if necessary, changing the signs of both sides. In the xy-plane (Fig. 2) mark the point P(a, b), and draw the circle

$$x^2 + y^2 = c^2.$$

If $a^2 + b^2 > c^2$, the point P lies outside the circle. From P draw tangents PQ and PR to the circle, and join OQ and OR. Then the angles XOQ and XOR are the solutions of (3). By drawing the figure and measuring these angles the solutions obtained by the methods of § 2 and § 3 can be verified.

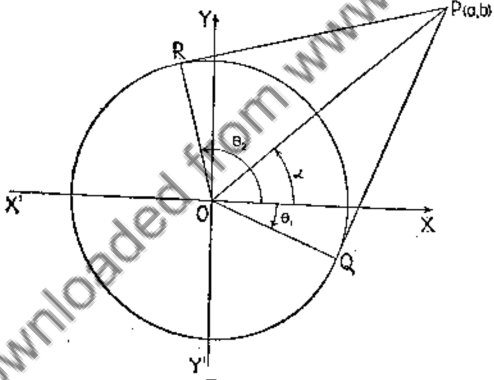

FIG. 2.

To prove this, let $\angle XOP = \alpha$, $\angle XOQ = \theta_1$, $\angle XOR = \theta_2$.

Then

$\text{OP} = \sqrt{(a^2 + b^2)}$,

$$\cos \alpha = a/\sqrt{(a^2 + b^2)}, \quad \sin \alpha = b/\sqrt{(a^2 + b^2)},$$

and OP makes angles $\alpha - \theta_1$, $\alpha - \theta_2$ with OQ and OR, respectively. Hence the projection of OP on OQ is

$$\text{OP} \cos (\alpha - \theta_1) = \text{OQ} = c\,;$$

so that

$$\sqrt{(a^2 + b^2)} \, . \, (\cos \alpha \cos \theta_1 + \sin \alpha \sin \theta_1) = c,$$

or

$$a \cos \theta_1 + b \sin \theta_1 = c$$

Thus $\angle$XOQ satisfies (3). Similarly $\angle$XOR satisfies (3).

If $a^2 + b^2 = c^2$, P lies on the circle, Q and R coincide with P, and the fundamental roots are equal.

If $a^2 + b^2 < c^2$, P lies within the circle, and the roots are not real.

Alternative Method.—This method depends on the theorem in analytical geometry that the linear equation $ax + by = c$ represents a straight line. In the (x, y) plane draw the circle $x^2 + y^2 = 1$, and the above straight line. If $c^2 < a^2 + b^2$, the line will cut the circle in distinct points P and Q; if $c^2 = a^2 + b^2$ the line will touch the circle; and if $c^2 > a^2 + b^2$ the line and the circle will not intersect. In the first case, let $\angle \text{XOP} = \theta_1$, $\angle \text{XOQ} = \theta_2$. Then P and Q are the points $(\cos \theta_1, \sin \theta_1)$ and $(\cos \theta_2, \sin \theta_2)$ respectively. But P and Q both lie on the line; hence

$$a \cos \theta_1 + b \sin \theta_1 = c,$$

and

$$a \cos \theta_2 + b \sin \theta_2 = c,$$

so that θ_1 and θ_2 are roots of equation (3). To verify the solutions, draw the figure and measure the angles XOP and XOQ.

EXAMPLES VIII

1. If $y = 6 \cos x + 8 \sin x + 5$, find the values of x between 0 and 360° for which y has (i) maximum, (ii) minimum, (iii) zero values.

Sketch the graph, and show by means of it that the equation

$$13x = 90(6 \cos x + 8 \sin x + 5),$$

where the angles are expressed in degrees, has only one root between 0 and 360°.

Ans. (i) 53° 8′, (ii) 233° 8′, (iii) 173° 8′, 293° 8′.

2. Express $12 \cos x + 5 \sin x$ in the form $\text{R} \cos (x + \theta)$, where R is positive.

Sketch the graph of $y = 12\cos x + 5\sin x - 7$ for

$$0 \leqq x \leqq 360°,$$

and find the maximum and the minimum values of y, and the corresponding values of x.

Ans. $R = 13$, $\theta = -22°\ 37'$; Max. 6, $x = 22°\ 37'$; Min. -20, $x = 202°\ 37'$.

3. Find the maximum and the minimum values of $3\cos x - 5\sin x$, and the corresponding values of x between $-360°$ and $+360°$.

Draw the graph of $y = 3\cos x - 5\sin x$ for the above range, taking as scales 1 inch $= 90°$ on the x-axis, and 1 inch $= 2$ units on the y-axis.

Ans. Max. $\sqrt{(34)} = 5{\cdot}83095$, $x = -59°\ 2'$, $300°\ 58'$; Min. $-\sqrt{(34)}$, $x = -239°\ 2'$, $120°\ 58'$.

4. Find the maximum and the minimum values of the function $11\sin x - 7\cos x$, and the corresponding values of x between 0 and 360°.

Ans. $\pm\sqrt{(170)} = \pm 13{\cdot}0384$, $x = 122°\ 28'$, $302°\ 28'$.

5. If $y = \sin x + 2\cos x$, express y in the form $R\cos(x - \alpha)$, and deduce the maximum and the minimum values of y, and the corresponding values of x for the range from 0 to 360°. Sketch the graph.

Ans. $R = \sqrt{5}$, $\alpha = 26°\ 34'$; Max. $\sqrt{5}$, Min. $-\sqrt{5}$; $26°\ 34'$, $206°\ 34'$.

6. Express $2\cos x + 3\sin x$ in the form $R\cos(x - \alpha)$. If the angles are measured in degrees, draw as accurately as you can the graph of $y = 2\cos x + 3\sin x$ for the range $x = -360°$ to $x = 360°$.

Obtain from your graph and by calculation the values of x within the range for which $y = 1{\cdot}5$.

Prove graphically that the equation

$$2\cos x + 3\sin x = 3 - \frac{x}{60}$$

has three real roots between 0 and 360°.

Ans. $R = \sqrt{(13)}$, $\alpha = 56°\ 19'$; $-238°\ 16'$, $-9°\ 6'$, $121°\ 44'$, $350°\ 54'$.

7. Express $4\cos x - 3\sin x$ in the form $R\cos(x + \theta)$.

If the angles are measured in degrees, draw an accurate graph of $y = 4\cos x - 3\sin x$ for the range from $x = -360°$ to $x = 360°$.

If $y = -2$, find from the graph the corresponding values of x for this range, and verify your results by solving the equation

$$4\cos x - 3\sin x + 2 = 0.$$

How many roots has the equation

$$4 \cos x - 3 \sin x = \frac{x}{10} ?$$

Ans. $5 \cos (x + 36° \ 52')$; $-283° \ 17'$, $-150° \ 27'$, $76° \ 43'$, $209° \ 33'$; one.

8. Find the values of x between 0 and 360° for which the function $6 \cos x + 9 \sin x + 7$ has (i) maximum, (ii) minimum, (iii) zero values.

Sketch the graph and show by means of it that the equation

$$x = 4(6 \cos x + 9 \sin x + 7),$$

where the angles are expressed in degrees, has only one root between 0 and 360.

Ans. (i) 56° 19′; (ii) 236° 19′; (iii) 186° 39′; 285° 59′.

9. Find all the angles between 0 and 360° which satisfy the following equations:

(i) $8 \cos \theta + \sin \theta + 4 = 0$. Ans. 126° 52′, 247° 22′.
(ii) $5 \cos \theta - 2 \sin \theta = 3$. Ans. 34° 21′, 282° 3′.
(iii) $3 \cos \theta + 7 \sin \theta + 5 = 0$. Ans. 295° 46′, 197° 50′.
(iv) $5 \cos \theta + 12 \sin \theta = 8$. Ans. 15° 22′, 119° 24′.
(v) $6 \cos \theta + 5 \sin \theta = 3$. Ans. 107° 13′, 332° 23′.
(vi) $5 \cos \theta - 7 \sin \theta = 4$. Ans. 7° 50′, 243° 15′.
(vii) $2 \cos \theta - 5 \sin \theta = 4$. Ans. 249° 46′, 333° 50′.
(viii) $7 \cos \theta - 5 \sin \theta = 3$. Ans. 34° 3′, 254° 53′.
(ix) $9 \cos \theta + 7 \sin \theta = 3$. Ans. 112° 37′, 323° 8′.
(x) $2 \sin \theta + 4 \cos \theta = 3$. Ans. 74° 26′, 338° 42′.
(xi) $2 \cos \theta + \sin \theta = 1$. Ans. 90°, 323° 8′.
(xii) $5 \sin \theta + 3 \cos \theta = 2$. Ans. 128° 58′, 349° 6′.
(xiii) $3 \sin \theta - 4 \cos \theta = -1$. Ans. 41° 36′, 244° 40′.
(xiv) $4 \sin \theta - 5 \cos \theta = 3$. Ans. 79° 16′, 203° 24′.
(xv) $2 \cos \theta = 3 \sin \theta + 1$. Ans. 17° 35′, 229° 47′.
(xvi) $3 \cos \theta - \sin \theta = \frac{1}{2}$. Ans. 62° 28′, 260° 40′.
(xvii) $4 \cos \theta - 3 \sin \theta = 2$. Ans. 29° 33′, 256° 43′.
(xviii) $4 \cos \theta - \sin \theta = 3$. Ans. 29° 17′, 302° 39′.
(xix) $5 \cos \theta + 3 \sin \theta + 4 = 0$. Ans. 164° 17′, 257° 39′.
(xx) $4 \cos \theta - 3 \sin \theta = 2{\cdot}25$. Ans. 26° 23′, 259° 53′.
(xxi) $5 \cos \theta - 3 \sin \theta = 3$. Ans. 28° 4′, 270°.
(xxii) $5 \cos \theta + 13 \sin \theta = 10$. Ans. 113° 5′, 24° 51′.

10. Find the maximum and the minimum values of the function $\cos^2 \theta - 2 \sin \theta \cos \theta - 2 \sin^2 \theta - 1$. Obtain also the values of θ between 0 and 360° for which the function has (i) its maximum value, (ii) its minimum value, and (iii) the value zero.

Ans. Max. $\frac{1}{2}\{\sqrt{(13)} - 3\}$; Min. $-\frac{1}{2}\{\sqrt{(13)} + 3\}$; (i) 163° 9′, 343° 9′, (ii) 73° 9′, 253° 9′, (iii) 0, 146° 19′, 180°, 326° 19′, 360°.

11. Find the turning values of the function

$$4\cos^2\theta - 3\sin\theta\cos\theta,$$

and the corresponding values of θ between 0 and 180°.

Ans. Max. $4\frac{1}{2}$, 161° 34′; Min. $-\frac{1}{2}$, 71° 34′.

12. If x and y are the maximum and the minimum values of $a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta$, respectively, show that

$$\text{(i) } x + y = a + b, \quad \text{(ii) } xy = ab - h^2.$$

13. Express $p\sin^2\theta + q\sin\theta\cos\theta + r\cos^2\theta$ in the form $A\cos(2\theta - \beta) + C$, where A, C and β are independent of θ.

Use the result to find the greatest and the least values of the function $8\cos^2\theta + 9\sin\theta\cos\theta - 4\sin^2\theta$; and the values of θ, between 0 and 180°, at which these occur. Find also an acute angle θ for which the function is equal to zero.

Ans. $A = \frac{1}{2}\sqrt{\{(p-r)^2+q^2\}}$, $C = \frac{1}{2}(p+r)$, $\beta = \tan^{-1}\{q/(r-p)\}$;
Max., 19/2, 18° 26′; Min., $-11/2$, 108° 26′; Zero, 71° 10′.

14. Find the maximum and the minimum values of the function $3\sin^2\theta + 4\sin\theta\cos\theta$, and the corresponding values of θ between 0 and 180°.

Tabulate, correct to two decimal places, the values of this function for values of θ from 0 to 180° by steps of 15°. Draw the graph of the function, and find from it two values of θ which satisfy the equation

$$6\sin^2\theta + 8\sin\theta\cos\theta = 5.$$

Ans. Max., 4, 63° 26′; Min., -1, 153° 26′; 30° 13′, 96° 38′.

15. In a triangle ABC the angle B is a right angle; AB and BC are 4 inches and 3 inches long respectively. The circle with centre A and radius 4·5 inches cuts the circumscribed circle of the triangle in P and Q. Show that the angles BAP and BAQ are solutions of the equation $8\cos\theta + 6\sin\theta = 9$, and evaluate these angles by solving the equation. From your diagram show that the angles BAP and BAQ are

$$\cos^{-1}(\tfrac{4}{5}) \pm \cos^{-1}(\tfrac{9}{10}).$$

Ans. 11° 1′, 62° 43′.

16. AB is a fixed straight line of length a, and AC makes with AB a variable acute angle θ; D is the projection of B on AC, and E is the projection of D on AB. Express AE + ED in terms of θ; then find its maximum value, and the corresponding value of θ.

Ans. $\frac{1}{2}(\sqrt{2} + 1)a$, $22\frac{1}{2}$°.

17. Prove that $\alpha + \beta$ and $\alpha - \beta$ are roots of the equation $12\cos\theta + 5\sin\theta = 10$, where $\tan\alpha = 5/12$ and $\cos\beta = 10/13$, α and β being angles in the first quadrant. Evaluate these roots.

Prove also the following construction for solving the equation graphically. O is the origin, C the point (12, 5), and on OC as diameter a circle is described. With centre O and radius 10 a circle is drawn to cut the first circle in P and Q. Then the angles XOP, XOQ are the required solutions.
Ans. $\alpha = 22° 37'$, $\beta = 39° 43'$, $\alpha + \beta = 62° 20'$, $\alpha - \beta = -17° 6'$.

18. Find solutions between 0 and 360° for

$$a \sin \theta + b \cos \theta + c = 0,$$

given that $a = 5$, $b = -7$, $c = -6$. Deduce the solutions when $a = 5$, $b = 7$, $c = -6$.

Ans. 98° 42′, 190° 14′ ; 81° 18′, 349° 46′.

19. Find all the solutions between 0 and 180° of the equation

$$\sin \theta - 2 \sin 2\theta \cos \theta + \cos 3\theta = (\sqrt{3} - 1) \sin 3\theta + 1.$$

Ans. 0, 80°, 120°.

20. Find all the values of θ between 0 and 180° which satisfy the equation

$$\cos 2\theta + 3 \sin 2\theta = 2.$$

Ans. 10° 24′, 61° 10′.

21. Solve the equation

$$12 + 9 \cos 3\theta = 8 \sin 3\theta$$

for values of θ between $-180°$ and 180°.
Ans. $-75° 28'$, $-72° 17'$, 44° 32′, 47° 43′, 164° 32′, 167° 43′.

22. Solve the equation $\cos \frac{1}{2}\theta + 2 \sin \frac{1}{2}\theta = 2$ for values of θ between 0 and 360°.

Ans. 73° 44′, 180°.

23. Find all the values of θ between 0 and 360° which satisfy the equation

$$(1 - \tan \theta)(1 + \sin 2\theta) = (1 + \tan \theta)(1 - \cos 2\theta).$$

Ans. 30°, 135°, 150°, 210°, 315°, 330°.

24. Show that, if $a \cos \theta + b \sin \theta = R \cos (\theta - \alpha)$, then

$$b \cos \theta - a \sin \theta = - R \sin (\theta - \alpha),$$

and hence find all the values of θ between 0 and 360° which satisfy the equation

$$2 \cos \theta + 3 \sin \theta = (3 \cos \theta - 2 \sin \theta)^2.$$

Ans. 26° 53′, 85° 45′.

25. Prove that, if h is the altitude from A of the triangle ABC, and if the notation of Ch. IX, § 1 is employed,

$$2h \sin A - a \cos A = a \cos (B - C).$$

Hence, find the angles of the triangle if $a = 8$, $h = 3$, $\cos (B - C) = \frac{1}{4}$.

Ans. A = 64° 40′, B = 95° 25′, C = 19° 55′.

26. If α and β are roots of the equation $a \cos\theta + b \sin\theta = c$, which do not differ by a multiple of 2π, show that

$$\frac{\cos\frac{1}{2}(\alpha+\beta)}{a} = \frac{\sin\frac{1}{2}(\alpha+\beta)}{b} = \frac{\cos\frac{1}{2}(\alpha-\beta)}{c}.$$

27. If α, β are values of θ which satisfy the equation

$$a \tan\theta + b \sec\theta = c,$$

and whose difference is not a multiple of 2π, show that

$$\frac{\cos(\alpha+\beta)}{c^2 - a^2} = \frac{\cos(\alpha-\beta)}{2b^2 - c^2 - a^2} = \frac{1}{c^2 + a^2}.$$

28. Find by inspection one of the solutions of the equation $12\cos\theta - 5\sin\theta = 12$, and use the formula (*Example* 26)

$$\tan\tfrac{1}{2}(\alpha+\beta) = \frac{b}{a}$$

to obtain a second solution.

Ans. 0, 314° 46′.

29. If α and β are distinct solutions of the equation

$$a\cos\theta + b\sin\theta = c$$

between 0 and 2π, and if $\alpha + \beta$ also satisfies the equation, show that $a = c$.

30. Find the tangents of the angles that satisfy the equation

$$(m+2)\sin\theta + (2m-1)\cos\theta = 2m+1.$$

Ans. $4/3$, $2m/(m^2-1)$.

31. If

$$(m-n)\cos\theta - (mn+1)\sin\theta + (m+n) = 0,$$

show that $\tan\theta = \dfrac{2m}{1-m^2}$ or $\dfrac{2n}{n^2-1}$.

32. Express $(1+\sin\theta)(3\sin\theta + 4\cos\theta + 5)$ in terms of $\tan\frac{1}{2}\theta$, show that it is a perfect square, and hence solve the equation

$$(1+\sin\theta)(3\sin\theta + 4\cos\theta + 5) = 9,$$

for values of θ from 0 to 360°.

Ans. 0, 126° 52′, 360°.

33. A directed segment OA, of length 12, makes a positive acute angle θ with OX; and a segment OB, of length 7, makes a positive right angle with OA. If the projection of AB on OX is -8, find the projection of AB on OY.

Ans. $\theta = 85° \, 6'$; Proj. on OY $= 7\cos\theta - 12\sin\theta = -11{\cdot}358$.

CHAPTER IX

TRIANGLE FORMULÆ

§ 1. Notation

In the formulæ for a triangle ABC the magnitudes of the angles BAC, CBA, ACB will be denoted by A, B, C, and the lengths of the opposite sides BC, CA, AB by a, b, c, respectively. The lengths of the radii of the circum-circle, the inscribed circle, and the escribed circles opposite A, B, C will be denoted respectively by R, r, r_1, r_2, r_3, and the centres of these circles by O, I, I_1, I_2, I_3. The area of the triangle will be represented by Δ, and the semi-perimeter by s, so that

$$2s = a + b + c.$$

§ 2. Relations between the Sides and the Angles of a Triangle

Three important sets of formulæ for the triangle will now be established. The first set is

$$\left.\begin{aligned} a &= b\cos C + c\cos B \\ b &= c\cos A + a\cos C \\ c &= a\cos B + b\cos A \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

The second, called *the law of sines*, is

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \quad . \quad . \quad (2a)$$

or, more definitely

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R. \quad . \quad . \quad (2)$$

The third set is known as *the law of cosines*, each equation may be written in two different forms which are both useful. They are

$$\left.\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A, \quad \text{or} \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc} \\ b^2 &= c^2 + a^2 - 2ca\cos B, \quad \text{or} \quad \cos B = \frac{c^2 + a^2 - b^2}{2ca} \\ c^2 &= a^2 + b^2 - 2ab\cos C, \quad \text{or} \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab} \end{aligned}\right\} \quad . \quad (3)$$

Each of the three sets of formulæ will be established independently. It will, however, be shown that, if any one of the three sets (1), (2a), and (3) be taken as fundamental, the other two sets can be derived from it.

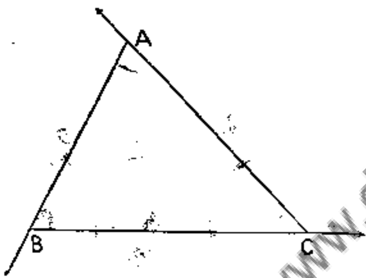

FIG. 1.

Set (1) can be proved by projection. Let BC, CA, AB (Fig. 1) be taken as the positive directions of the sides of the triangle. Then CA and AB make angles $(180° - C)$ and $(180° - A)$ with BC and CA respectively; hence AB makes the angle $(360° - C - A)$ with BC. Now project the three sides on BC, and get

$$a + b\cos(180° - C) + c\cos(360° - C - A) = 0,$$

or $$a - b\cos C + c\cos(C + A) = 0,$$

or $$a - b\cos C + c\cos(180° - B) = 0,$$

or, finally, $$a = b\cos C + c\cos B.$$

The second and third of equations (1) are obtained by projecting on CA and AB.

Example 1.—By projecting on a line perpendicular to BC, prove that

$$b\sin C = c\sin B.$$

In order to deduce equation (2a) from (1), substitute for b from the second equation of (1) into the first: thus

$$a = (c\cos A + a\cos C)\cos C + c\cos B.$$

Hence

$$a(1 - \cos^2 C) = c\{\cos A \cos C - \cos (A + C)\}$$

so that $\quad a \sin^2 C = c \sin A \sin C,$

whence $$\frac{a}{\sin A} = \frac{c}{\sin C}.$$

The other equations of (2a) can be deduced in the same way.

In order to deduce equations (3), multiply equations (1) by a, b, c respectively; thus

$$a^2 = ab \cos C + ca \cos B,$$
$$b^2 = bc \cos A + ab \cos C,$$
$$c^2 = ca \cos B + bc \cos A.$$

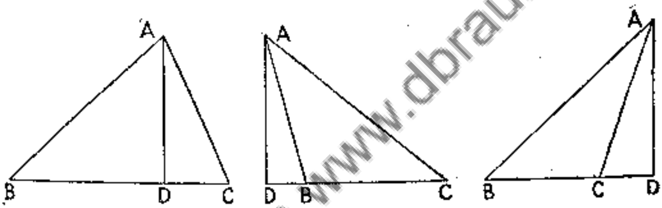

FIG. 2.

Now subtract the first equation from the sum of the other two, and so get

$$b^2 + c^2 - a^2 = 2bc \cos A,$$

the first of equations (3). The others are obtained in the same way.

The Law of Sines.—A proof of equations (2a) independent of (1) and (3) will now be given. Draw AD (Fig. 2) perpendicular to BC or BC produced. Then, if d is the length of DA,

$$d = c \sin B = b \sin C;$$

so that $$\frac{b}{\sin B} = \frac{c}{\sin C}.$$

The other equations of set (2a) can be obtained by employing the perpendicular from B to CA or the perpendicular from C to AB.

In order to complete the proof of (2) by showing that each ratio in ($2a$) is equal to $2R$, it should be noted that at least two of the angles of the triangle ABC must be acute. Suppose that A is acute, and draw a diameter BOD (Fig. 3a) of the circum-circle; join CD. Then

$$\angle BDC = \angle BAC = A,$$

and $\angle DCB$ is a right angle. Hence

$$BC = BD \sin A,$$

or

$$a = 2R \sin A, \qquad . \qquad . \qquad . \qquad (4)$$

from which, along with ($2a$), equations (2) are obtained.

Note.—Formula (4) can be established when A is obtuse (Fig. 3b), by taking on the circle a point A′ on the opposite

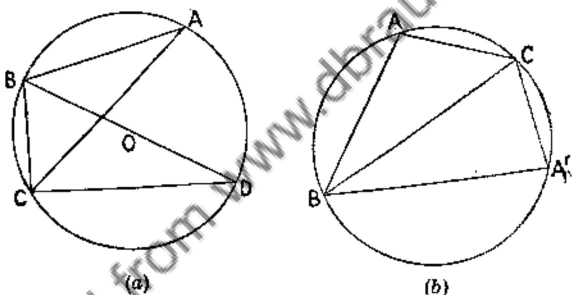

FIG. 3.

side of BC from A. Then since, in triangle A′BC, the angle A′ is acute

$$a = 2R \sin A' = 2R \sin (180° - A) = 2R \sin A.$$

If A is a right angle, BC is a diameter, and therefore $a = 2R$; also $\sin A = 1$, so that the formula holds in this case also. Thus (4) and consequently (2) can be established without using ($2a$).

In order to deduce (1) from ($2a$), let

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \lambda.$$

Then

$$\begin{aligned} b \cos C + c \cos B &= \lambda (\sin B \cos C + \sin C \cos B) \\ &= \lambda \sin (B + C) = \lambda \sin A = a. \end{aligned}$$

To obtain (3) multiply the identity of Chapter VII, § 3, *Example* 3, by λ^2.

The Law of Cosines.—Formulæ (3) can be derived directly from two well-known theorems in geometry.

Let AD be the altitude of the triangle ABC. Then (Fig. 4*a*) if the angle C is acute,

$$\begin{aligned} AB^2 &= BC^2 + CA^2 - 2BC \,.\, DC \\ &= BC^2 + CA^2 - 2BC \,.\, CA \cos C; \end{aligned}$$

while (Fig. 4*b*) if the angle C is obtuse,

$$\begin{aligned} AB^2 &= BC^2 + CA^2 + 2BC \,.\, CD \\ &= BC^2 + CA^2 - 2BC \,.\, CA \cos C. \end{aligned}$$

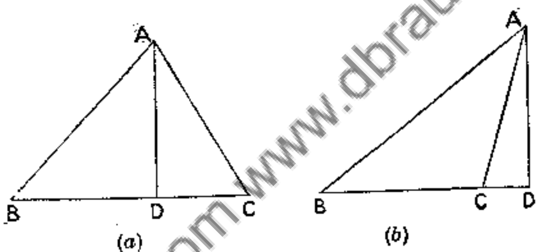

FIG. 4.

Hence, in each case,

$$c^2 = a^2 + b^2 - 2ab \cos C,$$

the third of equations (3). If C is a right angle, $\cos C = 0$, and the formula is simply a statement of the Theorem of Pythagoras. The other equations of (3) can be derived in the same way.

To deduce (1) from (3), add the second and third equations of (3): thus

$$b^2 + c^2 = c^2 + 2a^2 + b^2 - 2ca \cos B - 2ab \cos C.$$

Now cancel b^2 and c^2, and divide by $2a$, so obtaining the first of equations (1). Formulæ (2*a*) can then be deduced from formulæ (1).

Example 2.—Show that

$$\sin^2 A = \frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2}.$$

$$\left[\sin^2 A = 1 - \cos^2 A = 1 - \left(\frac{b^2+c^2-a^2}{2bc}\right)^2\right.$$

$$= \frac{\{2bc + (b^2+c^2-a^2)\}\{2bc - (b^2+c^2-a^2)\}}{4b^2c^2}$$

$$\left.= \frac{\{(b+c)^2 - a^2\}\{a^2 - (b-c)^2\}}{4b^2c^2}.\right]$$

Example 3.—Show that

$$\cos C = \frac{\sin^2 A + \sin^2 B - \sin^2 C}{2 \sin A \sin B}.$$

Example 4.—If

$$\frac{\sin A}{3} = \frac{\sin B}{3} = \frac{\sin C}{4},$$

prove that $\cos C = \frac{1}{9}$.

§ 3. Circular Functions of the Half-angles of a Triangle

Since

$$\cos A = \frac{b^2+c^2-a^2}{2bc} \quad \text{and} \quad \cos A = 2\cos^2 \tfrac{1}{2}A - 1,$$

$$2\cos^2 \tfrac{1}{2}A = 1 + \frac{b^2+c^2-a^2}{2bc} = \frac{(b+c)^2 - a^2}{2bc}$$

$$= \frac{(b+c+a)\,(b+c-a)}{2bc} = \frac{2s\,.\,2(s-a)}{2bc}.$$

Hence $$\cos \tfrac{1}{2}A = \sqrt{\left\{\frac{s(s-a)}{bc}\right\}}, \qquad (5a)$$

the positive sign for the square root being taken since the angle $\frac{1}{2}A$ is acute. Similarly it can be shown that

$$\cos \tfrac{1}{2}B = \sqrt{\left\{\frac{s(s-b)}{ca}\right\}} \qquad (5b)$$

and $$\cos \tfrac{1}{2}C = \sqrt{\left\{\frac{s(s-c)}{ab}\right\}}. \qquad (5c)$$

Again, since

$$\cos A = 1 - 2\sin^2 \tfrac{1}{2}A,$$

$$2\sin^2 \tfrac{1}{2}A = 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{a^2 - (b-c)^2}{2bc}$$

$$= \frac{(a - b + c)(a + b - c)}{2bc} = \frac{2(s-b) \cdot 2(s-c)}{2bc}.$$

Therefore $$\sin \tfrac{1}{2}A = \sqrt{\left\{\frac{(s-b)(s-c)}{bc}\right\}} \quad . \quad . \quad (6a)$$

Similarly $$\sin \tfrac{1}{2}B = \sqrt{\left\{\frac{(s-c)(s-a)}{ca}\right\}} \quad . \quad . \quad (6b)$$

and $$\sin \tfrac{1}{2}C = \sqrt{\left\{\frac{(s-a)(s-b)}{ab}\right\}} \quad . \quad . \quad (6c)$$

From formulæ (5) and (6) it follows by division that

$$\tan \tfrac{1}{2}A = \sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}}, \quad . \quad . \quad (7a)$$

$$\tan \tfrac{1}{2}B = \sqrt{\left\{\frac{(s-c)(s-a)}{s(s-b)}\right\}} \quad . \quad . \quad (7b)$$

and $$\tan \tfrac{1}{2}C = \sqrt{\left\{\frac{(s-a)(s-b)}{s(s-c)}\right\}} \quad . \quad . \quad (7c)$$

Again, from (5a) and (6a)

$$\sin A = 2\sin \tfrac{1}{2}A \cos \tfrac{1}{2}A$$

$$= 2\sqrt{\left\{\frac{(s-b)(s-c)}{bc}\right\}} \cdot \sqrt{\left\{\frac{s(s-a)}{bc}\right\}},$$

so that $$\sin A = \frac{2}{bc}\sqrt{\{s(s-a)(s-b)(s-c)\}} \quad . \quad (8a)$$

Similarly $$\sin B = \frac{2}{ca}\sqrt{\{s(s-a)(s-b)(s-c)\}} \quad . \quad (8b)$$

and $$\sin C = \frac{2}{ab}\sqrt{\{s(s-a)(s-b)(s-c)\}} \quad . \quad (8c)$$

§ 4. Area of a Triangle

Since the area $\triangle$ of the triangle ABC is

$$\tfrac{1}{2} \text{ base} \times \text{altitude},$$

and the altitude is $b \sin C$ or $c \sin B$,

$$\triangle = \tfrac{1}{2}ab \sin C = \tfrac{1}{2}ac \sin B.$$

The area is therefore equal to half the rectangle contained by two adjacent sides, multiplied by the sine of the included angle: thus

$$\triangle = \tfrac{1}{2}bc \sin A = \tfrac{1}{2}ca \sin B = \tfrac{1}{2}ab \sin C. \qquad (9)$$

Hence, from (8),

$$\triangle = \sqrt{\{s(s-a)(s-b)(s-c)\}}. \qquad (10)$$

Example 1.—If ABC is an acute-angled triangle, prove that the lengths of the sides and the magnitudes of the angles of the pedal triangle are

$$a \cos A,\ b \cos B,\ c \cos C,\ 180° - 2A,\ 180° - 2B,\ 180° - 2C.$$

Deduce that the area of the pedal triangle is

$$2\triangle \cos A \cos B \cos C.$$

[If AD, BE, CF are the perpendiculars from the vertices A, B, C of the triangle ABC on the opposite sides, DEF is the pedal triangle.]

Example 2.—If the angle A of the triangle ABC is obtuse, show that the sides and angles of the pedal triangle are

$$-a \cos A,\ b \cos B,\ c \cos C,\ 2A - 180°,\ 2B,\ 2C,$$

and that its area is $-2\triangle \cos A \cos B \cos C$.

Example 3.—If α, β, γ are the lengths of the medians of the triangle ABC, show that

$$\triangle = \tfrac{4}{3}\sqrt{\{\sigma(\sigma-\alpha)(\sigma-\beta)(\sigma-\gamma)\}},$$

where $2\sigma = \alpha + \beta + \gamma$.

[Let the medians AA′, BB′, CC′ meet in G. Produce GB′ its own length to L, and apply formula (10) to the triangle AGL.]

Example 4.—If P and Q are positive acute angles such that P > Q, deduce the formulæ

$$\text{(i)} \quad \sin(P+Q) = \sin P \cos Q + \cos P \sin Q,$$
$$\text{(ii)} \quad \sin(P-Q) = \sin P \cos Q - \cos P \sin Q$$

from formula (9).

On a straight line AD (Fig. 5, a) construct angles BAD and DAC equal to P and Q respectively, and draw BDC perpendicular to AD to meet AB and AC in B and C. Then

$$\begin{aligned}\tfrac{1}{2}AB \,.\, AC \sin (P + Q) &= \triangle ABC \\ &= \triangle ABD + \triangle ADC \\ &= \tfrac{1}{2}AB \,.\, AD \sin P + \tfrac{1}{2}AD \,.\, AC \sin Q.\end{aligned}$$

Hence
$$\begin{aligned}\sin (P + Q) &= \sin P \,.\, \frac{AD}{AC} + \frac{AD}{AB} \sin Q \\ &= \sin P \cos Q + \cos P \sin Q.\end{aligned}$$

Again, on AD (Fig. 5, *b*) construct angles DAC and DAB equal to P and Q respectively, and draw DBC perpendicular to AB. Then

$$\begin{aligned}\tfrac{1}{2}AB \,.\, AC \sin (P - Q) &= \triangle ABC \\ &= \triangle ADC - \triangle ADB \\ &= \tfrac{1}{2}AD \,.\, AC \sin P - \tfrac{1}{2}AD \,.\, AB \sin Q,\end{aligned}$$

from which (ii) follows.

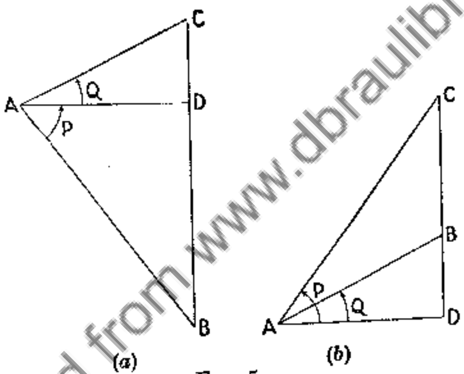

FIG. 5.

The corresponding formulæ for the cosine can easily be deduced.

Example 5.—If P and Q are the angles of *Example* 4, use formula (3) to prove that

(i) $\cos (P + Q) = \cos P \cos Q - \sin P \sin Q,$
(ii) $\cos (P - Q) = \cos P \cos Q + \sin P \sin Q.$

[In Fig. 5, *a*

$$\begin{aligned}\cos (P + Q) &= \frac{AB^2 + AC^2 - BC^2}{2AB \,.\, AC} \\ &= \frac{AB^2 + AC^2 - (BD^2 + DC^2 + 2BD \,.\, DC)}{2AB \,.\, AC} \\ &= \frac{AD}{AB} \,.\, \frac{AD}{AC} - \frac{BD}{AB} \,.\, \frac{DC}{AC}.\end{aligned}$$
]

§ 5. Radii of the Circumscribed, Inscribed, and Escribed Circles

Since $$R = \frac{a}{2\sin A} = \frac{abc}{2bc\sin A},$$

and $$\Delta = \tfrac{1}{2}bc\sin A,$$

then $$R = \frac{abc}{4\Delta} \quad . \quad . \quad . \quad . \quad (11)$$

Example 1.—Prove that the radius of the circum-circle of the pedal triangle is $\frac{1}{2}R$. [Use (11) and *Examples* 1, 2 of § 4.]

Let AI, BI, CI (Fig. 6), the bisectors of the angles of

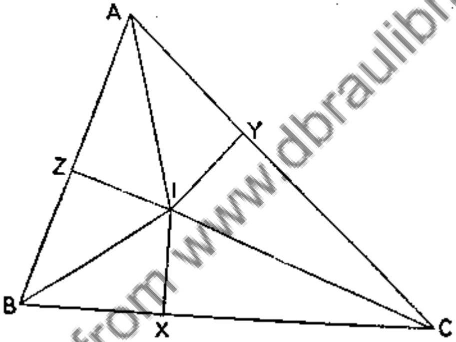

FIG. 6.

the triangle ABC, meet in I, the centre of the inscribed circle. Draw IX, IY, IZ perpendicular to BC, CA, AB respectively; each perpendicular is of length r. Then

$$\begin{aligned}\Delta &= \Delta IBC + \Delta ICA + \Delta IAB \\ &= \tfrac{1}{2}BC \,.\, IX + \tfrac{1}{2}CA \,.\, IY + \tfrac{1}{2}AB \,.\, IZ \\ &= \tfrac{1}{2}r(a+b+c) = rs.\end{aligned}$$

Hence $$r = \frac{\Delta}{s} \quad . \quad . \quad . \quad . \quad (12)$$

Example 2.—Prove that

$$r = 4R\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C.$$

Example 3.—Show that

$$\frac{AI \,.\, BI \,.\, CI}{abc} = \frac{r}{s}.$$

Again, if AI_1 (Fig. 7) is the internal bisector of the angle A, and BI_1, CI_1 are the external bisectors of the angles B and C, I_1 is the centre of the escribed circle opposite A. Draw I_1X_1, I_1Y_1, I_1Z_1 perpendicular to BC, CA, AB respectively. Then

$$I_1X_1 = I_1Y_1 = I_1Z_1 = r_1.$$

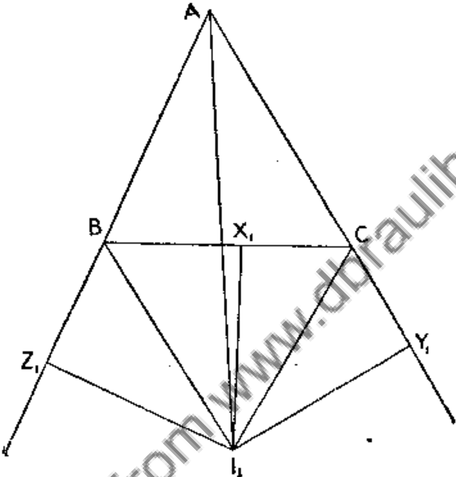

FIG. 7.

Hence

$$\begin{aligned}\Delta &= \triangle I_1CA + \triangle I_1AB - \triangle I_1CB\\ &= \tfrac{1}{2}r_1(b + c - a) = r_1(s - a).\end{aligned}$$

Therefore $$r_1 = \frac{\Delta}{s - a} \quad . \quad . \quad . \quad (13a)$$

Similarly $$r_2 = \frac{\Delta}{s - b} \quad . \quad . \quad . \quad (13b)$$

and $$r_3 = \frac{\Delta}{s - c} \quad . \quad . \quad . \quad (13c)$$

Again, in Fig. 6,

$$\begin{aligned}AB + AC &= AZ + ZB + AY + YC\\ &= AZ + BX + AY + XC\\ &= 2AZ + BC.\end{aligned}$$

Hence

$$2AZ = AB + AC - BC = c + b - a = 2(s - a),$$

so that $$AZ = s - a.$$

Thus, from triangle AZI,

$$\tan Z\widehat{A}I = \frac{IZ}{AZ},$$

and therefore $$\tan \tfrac{1}{2}A = \frac{r}{s-a} \quad . \quad . \quad . \quad (14a)$$

Similarly $$\tan \tfrac{1}{2}B = \frac{r}{s-b} \quad . \quad . \quad . \quad (14b)$$

and $$\tan \tfrac{1}{2}C = \frac{r}{s-c} \quad . \quad . \quad . \quad (14c)$$

Again (Fig. 7)

$$AZ_1 = AY_1 = \tfrac{1}{2}(AZ_1 + AY_1) = \tfrac{1}{2}(AB + BZ_1 + AC + CY_1)$$
$$= \tfrac{1}{2}(AB + BX_1 + AC + X_1C) = s.$$

Hence, from triangle I_1Z_1A, it follows that

$$\tan \tfrac{1}{2}A = \frac{r_1}{s}. \quad . \quad . \quad . \quad (15a)$$

Similarly $$\tan \tfrac{1}{2}B = \frac{r_2}{s} \quad . \quad . \quad . \quad (15b)$$

and $$\tan \tfrac{1}{2}C = \frac{r_3}{s}. \quad . \quad . \quad . \quad (15c)$$

Formulæ (14) and (15) can easily be deduced from (12) and (13) by means of (7) and (10).

Example 4.—Prove that

(i) $r_1 = 4R \sin \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C$,
(ii) $r_1 = a \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C \sec \tfrac{1}{2}A$,
(iii) $r_1 = (s - c) \cot \tfrac{1}{2}B = (s - b) \cot \tfrac{1}{2}C$.

§ 6. Napier's Tangent Formula

The formula

$$\tan \tfrac{1}{2}(B - C) = \frac{b-c}{b+c} \tan \tfrac{1}{2}(B + C) \quad . \quad (16)$$

is required for the solution of a triangle when two sides and the included angle are given. It may be proved as follows:

$$\frac{b-c}{b+c}=\frac{2R(\sin B-\sin C)}{2R(\sin B+\sin C)}=\frac{2\sin\frac{1}{2}(B-C)\cos\frac{1}{2}(B+C)}{2\sin\frac{1}{2}(B+C)\cos\frac{1}{2}(B-C)}$$
$$=\frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)},$$

from which, on multiplying by $\tan\frac{1}{2}(B+C)$, the formula is obtained. If $c > b$ it is more convenient to use the formula in the form

$$\tan\tfrac{1}{2}(C-B)=\frac{c-b}{c+b}\tan\tfrac{1}{2}(C+B).$$

By interchanging the letters in cyclical order other formulæ of the same kind are obtained.

Example 1.—Prove that

$$\text{(i)}\ \frac{a+b}{c}=\frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C},\quad \text{(ii)}\ \frac{a-b}{c}=\frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}.$$

Example 2.—The formula (16) can also be derived geometrically as follows. If $b > c$, draw a circle, with A as centre and AC as radius, to cut AB produced in D and BA produced in E; join CD and CE. Then $EB = b + c$, $BD = b - c$, $\angle CDB = \frac{1}{2}(B+C)$, $\angle BCD = \frac{1}{2}(B-C)$, $\angle BEC = 90^\circ - \frac{1}{2}(B+C)$, $\angle ECB = 90^\circ - \frac{1}{2}(B-C)$. Hence, from triangles BDC and EBC,

$$\frac{b-c}{a}=\frac{\sin\frac{1}{2}(B-C)}{\sin\frac{1}{2}(B+C)}\ \text{and}\ \frac{b+c}{a}=\frac{\cos\frac{1}{2}(B-C)}{\cos\frac{1}{2}(B+C)}.$$

Formula (16) can then be derived by division.

§ 7. Additional Properties of the Triangle ABC

The proofs of the results contained in the examples following are left as exercises to the reader.

Example 1.—Prove that

(i) $\angle OAI = \frac{1}{2}(C - B)$,
(ii) $IA = r \operatorname{cosec} \frac{1}{2}A = 4R \sin\frac{1}{2}B \sin\frac{1}{2}C$,
(iii) $OI^2 = R^2(1 - 8\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C)$,
(iv) $OI^2 = R^2 - 2Rr$.

[For (ii) use *Example* 2, § 5.]

Example 2.—Show that

(i) $AI_1 = 4R\cos\frac{1}{2}B\cos\frac{1}{2}C$,
(ii) $OI_1^2 = R^2(1 + 8\sin\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C)$,
(iii) $OI_1^2 = R^2 + 2Rr_1$.

[For (i) use *Example* 4, § 5.]

Example 3.—If H is the orthocentre of the triangle ABC, prove that

$$\text{(i)}\ AH = 2R\cos A,$$
$$\text{(ii)}\ \angle OAH = C - B,$$
$$\text{(iii)}\ OH^2 = R^2(1 - 8\cos A\cos B\cos C).$$

Example 4.—Prove that

$$\text{(i)}\ \angle IAH = \tfrac{1}{2}(C - B),$$
$$\text{(ii)}\ IH^2 = 2r^2 - 4R^2\cos A\cos B\cos C,$$
$$\text{(iii)}\ I_1H^2 = 2r_1^2 - 4R^2\cos A\cos B\cos C.$$

Example 5.—By applying formula (3) to the pedal triangle, show that

$$a^2\cos^2 A = b^2\cos^2 B + c^2\cos^2 C + 2bc\cos B\cos C\cos 2A.$$

Example 6.—If an angle of the triangle ABC is obtuse, prove that the triangle is self-polar with regard to a circle with H as centre (the polar circle). If ρ is the radius of this circle, prove that

$$\rho^2 = -4R^2\cos A\cos B\cos C.$$

Example 7.—If N is the mid-point of OH, prove that

$$\text{(i)}\ IN = \tfrac{1}{2}R - r,\quad \text{(ii)}\ I_1N = \tfrac{1}{2}R + r_1.$$

Hence show that the inscribed and escribed circles touch the nine-point circle; (Feuerbach's Theorem).

Example 8.—If K is a point on the base BC of the triangle ABC, and if BK : KC $= y : z$, while θ is the angle CKA, prove that

$$(y + z)\cot\theta = z\cot B - y\cot C.$$

$$\left[\frac{BK}{AB} = \frac{\sin\angle BAK}{\sin\theta} = \frac{\sin(\theta - B)}{\sin\theta},\right.$$

$$\frac{KC}{CA} = \frac{\sin\angle KAC}{\sin\theta} = \frac{\sin(\theta + C)}{\sin\theta}.$$

$$\left.\text{Hence}\quad \frac{y}{z} = \frac{\sin(\theta - B)}{\sin(\theta + C)}\frac{\sin C}{\sin B} = \frac{\cot B - \cot\theta}{\cot C + \cot\theta}.\right]$$

Example 9.—If M is the mid-point of the base BC, and if θ is the angle CMA, show that

$$\cot\theta = \tfrac{1}{2}(\cot B - \cot C).$$

Example 10.—If the internal and the external bisectors of the angle A meet BC in L and M respectively, prove that

$$\text{(i)}\ AL = \frac{2bc}{b + c}\cos\tfrac{1}{2}A,\quad \text{(ii)}\ AM = \frac{2bc}{|b - c|}\sin\tfrac{1}{2}A.$$

Example 11.—If the median AA′ of the triangle ABC makes an angle α with AB, prove that

$$\text{(i)}\ c \sin \alpha = b \sin (A - \alpha),$$
$$\text{(ii)}\ \cot \alpha = 2 \cot A + \cot B.$$

§ 8. Cauchy's Proof of the Addition Theorem

The following proof is based on formula (3) and the formula for the square of the distance between two points in co-ordinate geometry.

In the co-ordinate plane (Fig. 8) let $\angle XOP_1$ and $\angle XOP_2$ represent the angles θ_1 and θ_2 respectively, these angles being of any magnitude and of either sign. Let P_1 and P_2 be the points (x_1, y_1) and (x_2, y_2) respectively; or, in polar co-ordinates, (r_1, θ_1) and (r_2, θ_2). Then, from (3),

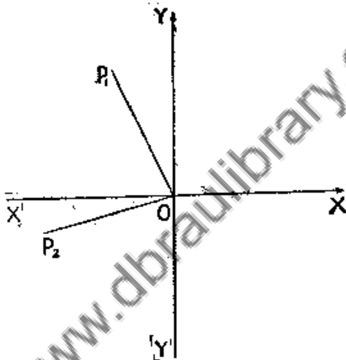

FIG. 8.

$$P_1P_2^2 = OP_1^2 + OP_2^2 - 2OP_1 \cdot OP_2 \cos P_2\widehat{O}P_1$$
$$= r_1^2 + r_2^2 - 2r_1r_2 \cos (\theta_1 - \theta_2).$$

It should be noted that this formula holds for all possible values, positive or negative, of the angle $\theta_1 - \theta_2$. For, if $\angle XOP_1$ and $\angle XOP_2$ are angles coterminal with θ_1 and θ_2, whose numerical difference is less than (or equal to) 180°,

$$\theta_1 = \angle XOP_1 + m \cdot 360°, \quad \theta_2 = \angle XOP_2 + n \cdot 360°,$$

where m and n are integers; and therefore

$$\theta_1 - \theta_2 = \angle XOP_1 - \angle XOP_2 + (m - n) \cdot 360°$$
$$= \angle P_2OP_1 + (m - n) \cdot 360°.$$

Thus $\qquad \cos (\theta_1 - \theta_2) = \cos P_2\widehat{O}P_1.$

Now

$$P_1P_2^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$
$$= (x_1^2 + y_1^2) + (x_2^2 + y_2^2) - 2x_1x_2 - 2y_1y_2$$
$$= r_1^2 + r_2^2 - 2r_1 \cos \theta_1 \cdot r_2 \cos \theta_2 - 2r_1 \sin \theta_1 \cdot r_2 \sin \theta_2.$$

Hence, on comparing the two expressions for $P_1P_2^2$, we see that

$$\cos (\theta_1 - \theta_2) = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2.$$

As the angles θ_1 and θ_2 are quite general, the formulæ for $\cos(\theta_1 + \theta_2)$, $\sin(\theta_1 + \theta_2)$ and $\sin(\theta_1 - \theta_2)$ can be deduced from this formula.

EXAMPLES IX

In the following examples the identities are to be established for any triangle ABC:

1. $a \sin A - b \sin B = c \sin(A - B)$.
2. $a \sin A + b \sin B - c \sin C = 2a \sin B \cos C$.
3. $a^2 \sin 2B + b^2 \sin 2A = 2ab \sin C = 4\Delta$.
4. $a^2 \sin 2B - b^2 \sin 2A = 2ab \sin(A - B)$.
5. $a \cos(B - C) = b \cos B + c \cos C$.
6. $a(b \cos C - c \cos B) = b^2 - c^2$.
7. (i) $b \sin(\theta + C) + c \sin(\theta - B) = a \sin\theta$;
   (ii) $b \cos(\theta + C) + c \cos(\theta - B) = a \cos\theta$,

where θ is any angle.

8. $c \sin(A - B) + a \sin(B - C) + b \sin(C - A) = 0$.
9. $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C = \dfrac{2\Delta}{R^2}$.
10. $\dfrac{\cos A}{\sin B \sin C} + \dfrac{\cos B}{\sin C \sin A} + \dfrac{\cos C}{\sin A \sin B} = 2$.
11. $a \cos A + b \cos B + c \cos C = 2\Delta/R$.
12. $a \cos(B - C) + b \cos(C - A) + c \cos(A - B) = \dfrac{abc}{R^2}$.
13. $\sin(120^\circ - A) + \sin(120^\circ - B) + \sin(120^\circ - C)$
    $= 4 \cos(60^\circ - \tfrac{1}{2}A) \cos(60^\circ - \tfrac{1}{2}B) \cos(60^\circ - \tfrac{1}{2}C)$.
14. $a^2 \sin(B - C) = (b^2 - c^2) \sin A$.
15. $\dfrac{a^2 \sin(B - C)}{\sin B + \sin C} + \dfrac{b^2 \sin(C - A)}{\sin C + \sin A} + \dfrac{c^2 \sin(A - B)}{\sin A + \sin B} = 0$.
16. $\cos A + \cos(B - C) = \dfrac{2\Delta}{aR}$.
17. $\dfrac{\cos A}{a} + \dfrac{a}{bc} = \dfrac{\cos B}{b} + \dfrac{b}{ca} = \dfrac{\cos C}{c} + \dfrac{c}{ab}$.
18. $a^2 \cos(B - C) + (b^2 + c^2) \cos A = 2bc$.
19. $a(b \cos B - c \cos C) + (b^2 - c^2) \cos A = 0$.
20. $(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0$.
21. $\dfrac{a \cos B - b \cos A}{c} = \dfrac{\sin^2 A - \sin^2 B}{\sin^2 C}$.
22. $a^2 \cos^2 A - b^2 \cos^2 B - c^2 \cos^2 C$
    $= 2R^2 \cos 2A \sin 2B \sin 2C$.

23. $a^2 \cos^2 A + b^2 \cos^2 B + c^2 \cos^2 C$
$$= 2R^2(1 - \cos 2A \cos 2B \cos 2C).$$

24. $\cot A + \dfrac{\sin A}{\sin B \sin C} = \dfrac{a^2 + b^2 + c^2}{4\Delta}.$

25. $\cot A + \cot B + \cot C = \dfrac{a^2 + b^2 + c^2}{4\Delta}.$

26. $\cos^2 C = \sin^2 A + \cos^2 B - 2 \sin A \cos B \sin C.$

27. $a^2 \tan^2 B - 2ab \tan B \sin C + b^2 = b^2 \sec^2 B \cos^2 C.$

28. $a^3 \cos A + b^3 \cos B + c^3 \cos C$
$$= abc(1 + 4 \cos A \cos B \cos C).$$

29. $c^3 \cos (A - B) + a^3 \cos (B - C) + b^3 \cos (C - A)$
$$= 3abc.$$

30. $\dfrac{1 + \cos C \cos (A - B)}{1 + \cos B \cos (C - A)} = \dfrac{a^2 + b^2}{c^2 + a^2}.$

31. $\dfrac{\sin (A - B)}{\sin (B - C)} = \dfrac{a(a^2 - b^2)}{c(b^2 - c^2)}.$

32. $\dfrac{1}{c \sin A} + \dfrac{1}{a \sin B} + \dfrac{1}{b \sin C} = \dfrac{1}{r}.$

33. $a^2 = (b - c)^2 \cos^2 \tfrac{1}{2}A + (b + c)^2 \sin^2 \tfrac{1}{2}A.$

34. $a^2 = (b + c)^2 - 4bc \cos^2 \tfrac{1}{2}A.$

35. $a^2 = (b - c)^2 + 4bc \sin^2 \tfrac{1}{2}A.$

36. $a^2 = (b - c)^2 \cos^2 \tfrac{1}{2}A \sec^2 \phi$, where $\tan \phi = \dfrac{b + c}{b - c} \tan \tfrac{1}{2}A.$

37. $a = (b + c) \sin \phi$, where ϕ is acute and
$$\cos \phi = \frac{2\sqrt{(bc)}}{b + c} \cos \tfrac{1}{2}A.$$

38. $a^2 = (b - c)^2 \sec^2 \phi$, where $\tan \phi = \dfrac{2\sqrt{(bc)}}{b - c} \sin \tfrac{1}{2}A.$

39. $s = 4R \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C.$

40. $s = r \cot \tfrac{1}{2}A \cot \tfrac{1}{2}B \cot \tfrac{1}{2}C.$

41. $a = r(\cot \tfrac{1}{2}B + \cot \tfrac{1}{2}C).$

42. $s = r (\cot \tfrac{1}{2}A + \cot \tfrac{1}{2}B + \cot \tfrac{1}{2}C).$

43. $\Delta = r^2 \cot \tfrac{1}{2}A \cot \tfrac{1}{2}B \cot \tfrac{1}{2}C$
$$= s^2 \tan \tfrac{1}{2}A \tan \tfrac{1}{2}B \tan \tfrac{1}{2}C.$$

44. $\Delta = 4Rr \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C.$

45. $s - a = 4R \cos \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C.$

46. $s \sin \tfrac{1}{2}A = a \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C.$

47. $(s - a) \sin \tfrac{1}{2}A = a \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C.$

48. $\tan \frac{1}{2}B \tan \frac{1}{2}C = \dfrac{s-a}{s}.$

49. $2r \cos \frac{1}{2}A + a \sin \frac{1}{2}A = a \cos \frac{1}{2}(B - C).$

50. $a \cot \frac{1}{2}A = (s - a)(\cot \frac{1}{2}B + \cot \frac{1}{2}C).$

51. $a \cot \frac{1}{2}A = s\,(\tan \frac{1}{2}B + \tan \frac{1}{2}C).$

52. $c \sin^2 \frac{1}{2}B + b \sin^2 \frac{1}{2}C = s - a.$

53. $(c \cos^2 \frac{1}{2}B + b \cos^2 \frac{1}{2}C)(c \sin^2 \frac{1}{2}B + b \sin^2 \frac{1}{2}C) = bc \cos^2 \frac{1}{2}A.$

54. $\dfrac{\tan \frac{1}{2}B + \tan \frac{1}{2}C}{\tan \frac{1}{2}B - \tan \frac{1}{2}C} = \dfrac{a}{b-c}.$

55. $\dfrac{\cot \frac{1}{2}B + \tan \frac{1}{2}C}{\cot \frac{1}{2}B - \tan \frac{1}{2}C} = \dfrac{b+c}{a}.$

56. $\dfrac{\cot \frac{1}{2}B - \cot \frac{1}{2}C}{\cot \frac{1}{2}C - \cot \frac{1}{2}A} = \dfrac{b-c}{c-a}.$

57. $\dfrac{1}{(s-a)(s-b)} + \dfrac{1}{(s-b)(s-c)} + \dfrac{1}{(s-c)(s-a)} = \dfrac{1}{r^2}.$

58. $\dfrac{1}{a} \cos^2 \frac{1}{2}A + \dfrac{1}{b} \cos^2 \frac{1}{2}B + \dfrac{1}{c} \cos^2 \frac{1}{2}C = \dfrac{s^2}{abc} = \dfrac{s}{4Rr}.$

59. $a^2 \sec^2 \frac{1}{2}A - b^2 \sec^2 \frac{1}{2}B = c(a - b) \operatorname{cosec}^2 \frac{1}{2}C.$

60. $a \sin \frac{1}{2}A + b \sin (B + \frac{3}{2}A) = c \sin (B + \frac{1}{2}A).$

61. $(b + c) \cos \frac{1}{2}C \sin (A + \frac{1}{2}C)$
$$= (a + b) \cos \tfrac{1}{2}A \sin (C + \tfrac{1}{2}A).$$

62. $(a^2 + 4bc \cos^2 \frac{1}{2}A) \tan^2 \frac{1}{2}(B - C)$
$$= (a^2 - 4bc \sin^2 \tfrac{1}{2}A) \tan^2 \tfrac{1}{2}(B + C).$$

63. $\cos \frac{1}{2}(B - C) = \dfrac{r + R(1 - \cos A)}{2R \sin \frac{1}{2}A}.$

64. $\cos A + \cos B + \cos C = 1 + \dfrac{r}{R}.$

65. $a^2 + b^2 - 2ab \cos (C + 60^\circ)$
$$= b^2 + c^2 - 2bc \cos (A + 60^\circ)$$
$$= c^2 + a^2 - 2ca \cos (B + 60^\circ).$$

66. If $\frac{1}{4} \sin A = \frac{1}{5} \sin B = \frac{1}{6} \sin C$, find the values of $\cos A$, $\cos B$, $\cos C$.

Ans. $\cos A = \frac{3}{4}$, $\cos B = \frac{9}{16}$, $\cos C = \frac{1}{8}$.

67. If $\cos A = k \cos B$, prove that
$$\tan^2 \tfrac{1}{2}(A + B) = \frac{(a + b)(1 - k)}{(a - b)(1 + k)},$$
and that
$$c^2 = (a^2 - b^2)\,\frac{a + bk}{a - bk}.$$

68. Prove that

(i) $\dfrac{\sin 2A - \sin 2B + \sin 2C}{\sin 2A + \sin 2B - \sin 2C} = \dfrac{\tan B}{\tan C}$;

(ii) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

Apply (i) and (ii) to show that, if

$$\tfrac{1}{3}\sin 2A = \tfrac{1}{4}\sin 2B = \tfrac{1}{5}\sin 2C,$$

then $\tan A = 3, \quad \tan B = 2, \quad \tan C = 1.$

69. If $\tfrac{1}{3}\tan A = \tfrac{1}{4}\tan B = \tfrac{1}{5}\tan C = k$, prove that $k = 1/\sqrt{5}$, and find A, B, C correct to the nearest half-minute.

Ans. $A = 53° 18'$; $B = 60° 47\tfrac{1}{2}'$; $C = 65° 54\tfrac{1}{2}'$.

70. If $a + b = 3c$, prove that $\sin \tfrac{1}{2}C = \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B$.

71. $r_2 + r_3 = a \cot \tfrac{1}{2}A$.

72. $r_1 + r_2 + r_3 - r = 4R$.

73. $s^2 = r_1r_2 + r_2r_3 + r_3r_1$.

74. $\Delta = r_2r_3 \tan \tfrac{1}{2}A = \dfrac{r_1r_2r_3}{s}$.

75. $\dfrac{r_1}{r_2} + \dfrac{r_1}{r_3} = \dfrac{a}{s - a}$.

76. $r^3 = r_1r_2r_3 \tan^2 \tfrac{1}{2}A \tan^2 \tfrac{1}{2}B \tan^2 \tfrac{1}{2}C$.

77. $\tan \tfrac{1}{2}A + \tan \tfrac{1}{2}B + \tan \tfrac{1}{2}C = \dfrac{r_1 + r_2 + r_3}{\sqrt{(r_1r_2 + r_2r_3 + r_3r_1)}}$.

78. $\dfrac{b - c}{r_1} + \dfrac{c - a}{r_2} + \dfrac{a - b}{r_3} = 0$.

79. $\dfrac{1}{r} = \dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3}$.

80. $\Delta = \sqrt{(rr_1r_2r_3)}$.

81. $a = \dfrac{r_1(r_2 + r_3)}{\sqrt{(r_1r_2 + r_2r_3 + r_3r_1)}}$.

82. $4R = \dfrac{(r_1 + r_2)(r_2 + r_3)(r_3 + r_1)}{r_1r_2 + r_2r_3 + r_3r_1}$.

83. $\dfrac{r_1}{a}\cos^2 \tfrac{1}{2}A = \dfrac{r_2}{b}\cos^2 \tfrac{1}{2}B = \dfrac{r_3}{c}\cos^2 \tfrac{1}{2}C$

$$= \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C.$$

84. If $\sin^2 \tfrac{1}{2}A = \sin^2 \tfrac{1}{2}B + \sin^2 \tfrac{1}{2}C$, show that $r_1 = 2R$.

85. If $r_1 = R$, show that

$$\cos B + \cos C = \cos A = \frac{b + c}{a + b + c}.$$

86. Show that the triangle ABC is right angled if

$$r_1 = r_2 + r_3 + r.$$

87. Prove that $I I_1 = 4R \sin \frac{1}{2}A$, $I_1 I_2 = 4R \cos \frac{1}{2}C$, and $I_1 I_2^2 = 4R(r_1 + r_2)$.

88. Prove that

$$\text{Area } BI_1CI = 2R^2 \sin A(\cos B + \cos C)$$
$$= \frac{2a(b+c)\Delta}{(b+c+a)(b+c-a)}.$$

89. Prove that

$$\Delta I_1 I_2 I_3 = 2Rs.$$

90. If B = C, show that

$$r_2 = b \sin B = \tfrac{1}{2}a \tan B = b \cos \tfrac{1}{2}A = \sqrt{(b^2 - \tfrac{1}{4}a^2)}.$$

91. In a triangle ABC, $A = B = \theta$ radians, and AB = 2 inches. The escribed circle opposite A touches AC produced and AB produced at X and Y. Show that the area of the minor segment cut off by XY is

$$\tfrac{1}{2}\tan^2 \theta(\pi - \theta - \sin \theta) \text{ square inches.}$$

92. Prove that, in Fig. 7,

$$\Delta X_1 Y_1 Z_1 = 2r_1^2 \cos \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C.$$

93. Find the values of r, a, b, c, if $r_1 = 1$, $r_2 = 3$, $r_3 = 6$.

Ans. $r = \frac{2}{3}$; $a = \sqrt{3}$; $b = \frac{7}{3}\sqrt{3}$; $c = \frac{8}{3}\sqrt{3}$.

94. Find the sides and the angles of the triangle in which $r_1 = 7$, $r_2 = 10$, $r_3 = 3$.

Ans. $a = \frac{91}{11}$; $b = \frac{100}{11}$; $c = \frac{51}{11}$;
$A = 64° \, 56\frac{1}{2}'$; $B = 84° \, 33'$; $C = 30° \, 30\frac{1}{2}'$.

95. Prove that, if $\tan A = 1$ and $\tan B = 2$, then $\tan C = 3$.

96. If D, E are points on BC, and if the radii of the circles ABD, ABE, ADC, AEC are respectively R_1, R_2, R_3, R_4, show that $R_1 : R_2 = R_3 : R_4$.

97. If the altitude from A is equal to the radius of the circumcircle, prove that $2 \sin B \sin C = 1$.

98. If the altitude from A is half of BC, show that

$$\sin A - \cos A = \cos (B - C).$$

99. If B = 3C, show that

$$\cos C = \sqrt{\left(\frac{b+c}{4c}\right)}, \text{ and that } \sin \tfrac{1}{2}A = \frac{b-c}{2c}.$$

100. If B = 4A, show that

$$\cos A + \cos 3A = \frac{b}{2a}.$$

101. One side of a triangle is double another, and the angles opposite these sides differ by 60°. Prove that the triangle is right angled.

102. If $A = 30°$ and $CA = \frac{4}{3}AB$, calculate B and C.

Ans. $B = 38° 16'$; $C = 111° 44'$.

103. If $C = 60°$ and $b = 3a$, prove that $\tan A = \frac{1}{5}\sqrt{3}$.

104. The sides a, b, c are in arithmetic progression. Prove that

$$\cos \tfrac{1}{2}(C - A) = 2 \sin \tfrac{1}{2}B.$$

105. If $5(\cos C + \cos A) = 4(1 + \cos C \cos A)$, prove that

$$2b = c + a.$$

106. If $\tan \frac{1}{2}A$, $\tan \frac{1}{2}B$, $\tan \frac{1}{2}C$ are in harmonic progression, prove that a, b, c are in arithmetic progression.

107. If the perimeter is $3a$, show that

$$\tan \tfrac{1}{2}A \tan \tfrac{1}{2}B + \tan \tfrac{1}{2}C \tan \tfrac{1}{2}A = 2 \tan \tfrac{1}{2}B \tan \tfrac{1}{2}C = \tfrac{2}{3}.$$

108. If $\tan \frac{1}{2}A + \tan \frac{1}{2}B = k \cos \frac{1}{2}C$, show that

$$\tan \tfrac{1}{2}A \tan \tfrac{1}{2}B = 1 - k \sin \tfrac{1}{2}C.$$

109. If $\tan \frac{1}{2}A = p$ and $\tan \frac{1}{2}B = q$, show that

$$\sin C = \frac{2(p + q)(1 - pq)}{(1 + p^2)(1 + q^2)}.$$

110. Show that the area of a regular polygon of n sides is given by

$$nr^2 \tan \frac{\pi}{n}, \quad \text{and by} \quad \tfrac{1}{2}nR^2 \sin \frac{2\pi}{n},$$

where r, R are the radii of the inscribed and the circumscribed circles, respectively.

111. If BP is drawn perpendicular to BC to meet CA in P, prove that $CA : AP = \tan B : \tan C$.

112. If $AB = AC$, D is a point on AB and E a point on AC produced, such that $BD = CE$, prove trigonometrically that BC bisects DE.

113. If P is a point on the altitude AD such that $\angle CBP = \frac{1}{3}B$, prove that $AP = 2c \sin \frac{1}{3}B$.

114. If h_1, h_2, h_3 are the lengths of the altitudes from A, B, C, respectively, prove that

$$h_1 \cos A + h_2 \cos B + h_3 \cos C = \frac{a^2 + b^2 + c^2}{4R},$$

and that

$$\frac{1}{h_1} \cos A + \frac{1}{h_2} \cos B + \frac{1}{h_3} \cos C = \frac{1}{R}.$$

115. If the altitudes AD, BE, CF have lengths h_1, h_2, h_3, respectively, prove that

$$\frac{1}{r} = \frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3},$$

and that
$$\frac{1}{r_1} = -\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3}.$$

116. Equilateral triangles DBC, BEC are described on opposite sides of BC. Prove that

$$AD^2 + AE^2 = a^2 + b^2 + c^2.$$

117. If $A = 30°$ and $\Delta = \frac{\sqrt{3}}{4}a^2$, show that the other angles are 30° and 120°.

118. If $B > C$, and if the straight line joining the midpoint of the side AB to the foot of the altitude from A meets CA in F, prove that

$$AF = \frac{c \sin 2B}{2 \sin (B - C)}.$$

119. If D is a point on BC such that BD : DC $= m : n$, show that

$$m \cot DAB = (m + n) \cot A + n \cot B.$$

120. Points D, E are taken, on the side BC of a triangle ABC, such that BD = DE = EC. If $B\hat{A}D = x$, $D\hat{A}E = y$, $E\hat{A}C = z$, prove that

$$\frac{\sin (x + y) \sin (y + z)}{\sin x \sin z} = 4.$$

121. Points X and Y are taken, on BA produced and on AB produced, respectively, such that AX = AC and BY = BC; YP is drawn parallel to BC to meet XC in P. Prove that

$$YP = a\{1 + \sin \tfrac{1}{2}A/\cos \tfrac{1}{2}(B - C)\}.$$

122. If P is a point within the triangle ABC such that $\angle BAP = \angle CBP = \angle ACP = \theta$, prove that

$$PA = \frac{c \sin (B - \theta)}{\sin B} = \frac{b \sin \theta}{\sin A},$$

and that
$$\cot \theta = \cot A + \cot B + \cot C.$$

123. Find the other angles of a triangle in which one angle is 56° 26′, and the circumradius is three times the inradius.

Ans. 27° 24′; 96° 10′.

124. The perimeter of a triangle ABC is 50 inches, $A = 42°$ and $B = 68°$. Find the lengths of the sides.
Ans. $a = 13{\cdot}19$ inches; $b = 18{\cdot}28$ inches; $c = 18{\cdot}53$ inches.

125. In a triangle ABD, $\angle BAD = 35°$ and $\angle ABD = 90°$; AD is produced to C so that $AD = 2DC$. Find the angles of the triangle ABC, correct to the nearest minute.
Ans. $A = 35°$; $B = 115° \, 27'$; $C = 29° \, 33'$.

126. The diameter MN of a semicircle, whose centre is A, is produced to C, and C is joined to any point B on the semicircle. Prove that

$$\tan ABN : \tan NBC = MC : NC.$$

127. OAB is an equilateral triangle, and C is the midpoint of AB. On OC is described an equilateral triangle OCD, and AD meets OC in P and OB in Q. If $OA = 2$, prove that $AD = \sqrt{7}$, $\sin \angle OPA = 5/(2\sqrt{7})$ and

$$AP : PQ : QD = 6 : 4 : 5.$$

128. Prove that

$$1 < \cos A + \cos B + \cos C < 2.$$

[Apply *Example* 2, § 5.]

129. If $2R = 5r$, show that

$$\cos A + \cos B + \cos C = 1{\cdot}4.$$

130. If D, E, F are the feet of the altitudes, prove that the inradius of the triangle AFE is $r \cos A$, and that the sum of the inradii of the triangles AFE, BDF, CED is $r + r^2/R$.

131. B′, C′ are the midpoints of CA, AB, respectively, and O is the circumcentre of the triangle ABC. Show that the sides of the triangle OB′C′ are proportional to $\sin A$, $\cos B$, $\cos C$; and deduce that

$$1 - \cos^2 A - \cos^2 B - \cos^2 C - 2 \cos A \cos B \cos C = 0.$$

132. O is the circumcentre of a triangle ABC, and AO produced meets BC in P and the circumcircle in P′. Prove that $OP = R \cos A \sec (B - C)$, and that, if $OP = PP'$,

$$\tan B \tan C = 3.$$

133. The diameter of the circumcircle of a triangle ABC drawn from A meets BC in L. Prove that

$$\text{(i)} \ \frac{BL}{LC} = \frac{\sin 2C}{\sin 2B}; \quad \text{(ii)} \ BL = \frac{c \cos C}{\cos (B - C)}.$$

134. The diameters of the circumcircle of a triangle ABC drawn from A, B, C meet BC, CA, AB, respectively, in L, M, N. Prove that

$$\frac{1}{AL} + \frac{1}{BM} + \frac{1}{CN} = \frac{2}{R}.$$

135. If O is the circumcentre of a triangle ABC, and A′, B′, C′ are the midpoints of the sides, show that

$$OA' + OB' + OC' = R + r.$$

136. AL, BM, CN are diameters of the circumcircle of a triangle ABC. Prove that, due regard being paid to the sign of the area,

$$\triangle BLC = 2R^2 \sin A \cos B \cos C,$$

and that

$$\triangle BLC + \triangle CMA + \triangle ANB = \triangle ABC.$$

137. The bisectors of the angles A, B, C meet the circumcircle of the triangle ABC in L, M, N, respectively. Show that

$$\triangle BLC = R^2(\sin A - \tfrac{1}{2}\sin 2A),$$

and that the area of the hexagon BLCMAN is

$$4R^2 \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C.$$

138. The bisectors of the angles A, B, C meet the circumcircle of the triangle ABC in L, M, N, respectively. Show that $\triangle LMN = \frac{1}{2}Rs$, and that

$$\triangle LCM + \triangle MAN + \triangle NBL = \triangle LMN.$$

Show also that the perimeter of the triangle LMN is

$$8R \cos \tfrac{1}{4}(\pi - A) \cos \tfrac{1}{4}(\pi - B) \cos \tfrac{1}{4}(\pi - C),$$

and that, if the perimeters of the triangles ABC, LMN are equal

$$8 \cos \tfrac{1}{4}(\pi + A) \cos \tfrac{1}{4}(\pi + B) \cos \tfrac{1}{4}(\pi + C) = 1.$$

139. The bisector of the angle A of a triangle ABC meets BC in P, and PQ is drawn perpendicular to AC to meet AC in Q. Prove that

$$PQ = \frac{2R \sin A \sin B \sin C}{\sin B + \sin C}.$$

140. The bisector of the angle A meets BC in P and the circumcircle in K. Prove that

$$\frac{AP}{PK} = \frac{4s(s-a)}{a^2}.$$

Prove also that

$$PK = \frac{2R \sin^2 \frac{1}{2}A}{\cos \frac{1}{2}(B - C)}.$$

141. The bisector of the angle A makes an angle θ with BC. Prove that

$$\sin\theta = \cos\tfrac{1}{2}(B - C).$$

142. The internal bisector of the angle A meets BC in D. Prove that $AD = 2R \sin B \sin C / \cos\frac{1}{2}(B - C)$; and find an expression for AD in terms of a, b, c.

$$\text{Ans.}\quad \frac{\sqrt{\{bc(b+c+a)(b+c-a)\}}}{b+c}.$$

143. The inscribed circle touches the sides of the triangle ABC in X, Y, Z. Prove that the area of the triangle XYZ is $2r^2 \cos\frac{1}{2}A \cos\frac{1}{2}B \cos\frac{1}{2}C$.

144. Prove that the radius of the inscribed circle of the pedal triangle is $2R \cos A \cos B \cos C$.

145. Prove that the radius of the inscribed circle of the pedal triangle is $(R^2 - OH^2)/4R$.

Given the position of the circumcircle and the orthocentre of a triangle, prove that the vertices of its pedal triangle lie on a fixed circle and that the sides of that triangle touch a fixed circle.

146. If the inscribed circle touches BC, CA, AB in X, Y, Z, respectively, prove that a circle can be inscribed in the quadrilateral IYAZ, and that, if γ is its radius,

$$\frac{1}{\gamma} - \frac{1}{r} = \frac{1}{s-a}.$$

147. The part of the tangent, parallel to BC, to the incircle of a triangle ABC, which is intercepted between AB and AC, has length l. Similarly, m, n are the lengths of the parts of the tangents, parallel to CA, AB, respectively, which are intercepted between the other sides. Show that

$$\frac{l}{a} + \frac{m}{b} + \frac{n}{c} = 1.$$

148. If R_1, R_2, R_3 are the radii of the circles BIC, CIA, AIB, respectively, prove that

$$R_1 \operatorname{cosec}\tfrac{1}{2}A + R_2 \operatorname{cosec}\tfrac{1}{2}B + R_3 \operatorname{cosec}\tfrac{1}{2}C = 6R.$$

149. Show that, with the usual notation,

$$AH + BH + CH = 2(R + r).$$

150. With the usual notation for the triangle ABC, prove that

$$\text{(i)}\quad \frac{DH}{DA} = \cot B \cot C;$$

$$\text{(ii)}\quad \frac{DH}{DA} + \frac{EH}{EB} + \frac{FH}{FC} = 1.$$

151. If AD meets EF in K, prove that

$$\text{(i)}\ \frac{AK}{AH} = \frac{\sin B \sin C}{\cos (B - C)};$$

$$\text{(ii)}\ \frac{AK}{KH} = \frac{AD}{HD}.$$

152. Prove that

$$\text{(i)}\ DE = R \sin 2C;$$

$$\text{(ii)}\ \Delta DEF = \tfrac{1}{2}R^2 \sin 2A \sin 2B \sin 2C.$$

Show also that, if $\Delta DEH = \Delta DEF$, the angles A and B differ by a right angle.

153. If A′, B′, C′ are the mid-points of the sides, D, E, F, the feet of the altitudes, prove that $A'D = R \sin (C - B)$, and that

$$BC \cdot A'D + CA \cdot B'E + AB \cdot C'F = 0,$$

the directions BC, CA, AB being taken as positive.

154. Prove that $A'D = \tfrac{1}{2}a \dfrac{\tan C - \tan B}{\tan C + \tan B}$.

155. If $C = 2B$, prove trigonometrically that $CA = 2A'D$.

156. If x, y, z are the lengths of the altitudes of a triangle ABC, prove that $abcxyz = 8r^3s^3$.

157. The altitudes AD, BE, CF of a triangle ABC meet the circumcircle in P, Q, R, respectively. Prove that

$$DP = AD \cot B \cot C,$$

and that

$$\frac{DP}{AD} + \frac{EQ}{BE} + \frac{FR}{CF} = 1.$$

Prove also that

$$\Delta PCB + \Delta QAC + \Delta RBA = \Delta ABC.$$

158. In a triangle ABC, G is the centroid and I the incentre, and GI is parallel to BC. If PQ is drawn parallel to BC to touch the inscribed circle and meet AB, AC in P, Q, prove that (i) $PQ = \frac{1}{3}a$, (ii) the perimeter of the triangle APQ is $2(s-a)$, and (iii) $b + c = 2a$.

159. From G, the centroid of a triangle ABC, perpendiculars GP, GQ, GR are drawn to BC, CA, AB.

Prove that

$$\text{(i)}\ PG = \tfrac{1}{3}b \sin C,$$

$$\text{(ii)}\ \Delta PQR = \frac{abc(a^2 + b^2 + c^2)}{144R^3}.$$

160. If P is a point on a line drawn from A to cut BC internally, and if the perpendiculars from P to AB, AC have lengths h, k, prove that

$$AP \sin A = \sqrt{(h^2 + k^2 + 2hk \cos A)}.$$

161. If P is any point on the arc BC, opposite A, of the circle ABC, and PD, PE are the perpendiculars from P to AB, AC, show that DE = PA sin A, and that

$$PA \sin A = PB \sin B + PC \sin C.$$

162. If $b > c$ and if BM, CM are respectively perpendicular and parallel to the bisector of the angle A, prove that

$$BM = (b + c) \sin \tfrac{1}{2}A, \quad CM = (b - c) \cos \tfrac{1}{2}A.$$

163. Points X, Y are taken on the sides AB, CA, respectively, so that XY is parallel to BC and equal to BX + CY. Show that

$$XY = \frac{a(2s - a)}{2s} = \frac{a \cos \frac{1}{2}(B - C)}{2 \cos \frac{1}{2}B \cos \frac{1}{2}C}.$$

164. The tangents at A, B, C to the circumcircle of the triangle ABC are MN, NL, LM, respectively. Prove that the circumradius of the triangle LMN is $\frac{1}{2}$R sec A sec B sec C, and that MN = R sin A sec B sec C.

165. From a fixed point C, tangents CX, CY, of length t, are drawn to a fixed circle. The tangent at a variable point P on the circle cuts off, from CX, CY, lengths a, b, measured from C. Show that, for all positions of P on the circle, $(t - a)(t - b)/ab$ is constant.

166. On BC, CA, AB as bases, and outside the triangle ABC, isosceles triangles BPC, CQA, ARB, with base angles 30°, are described. Prove that the triangle PQR is equilateral.

167. In an acute angled triangle ABC, the circle on the altitude AD as diameter cuts AB in P and AC in Q. Show that

$$PQ = 2R \sin A \sin B \sin C = \frac{\Delta}{R}.$$

168. A, B, C are points in order on a straight line, and $AB = 2a$, $BC = 2b$, $b > a$. On AB and BC as diameters, circles are described, and a line is drawn from A, making an angle θ with ABC, to cut the smaller circle in P and the larger in Q, R. Show that, when AP = QR, $\cos 2\theta = \dfrac{5a - b}{3a + b}$.

169. In a triangle A′B′C′, $B'C' = b + c$, $C'A' = c + a$, $A'B' = a + b$, where a, b, c are the sides of a triangle ABC. Show that

$$\frac{\cot \frac{1}{2}A'}{\sin A} = \frac{\cot \frac{1}{2}B'}{\sin B} = \frac{\cot \frac{1}{2}C'}{\sin C}.$$

170. If BC = 5, CA = 4, AB = 3, and if D, E are points on BC, such that BD = DE = EC, prove that

$$\tan \text{CAE} = \tfrac{3}{8}, \quad \tan \text{BAD} = \tfrac{2}{3}, \quad \tan \text{DAE} = \tfrac{18}{25},$$

and find the angles of the triangle ADE.

Ans. $\angle \text{DAE} = 35° \, 45'$; $\angle \text{EDA} = 86° \, 49'$; $\angle \text{AED} = 57° \, 26'$.

171. The sides subtend equal angles at a point P within a triangle ABC. Prove that, if $AP = x$, $BP = y$, $CP = z$,

$$a^2(y - z) + b^2(z - x) + c^2(x - y) = 0.$$

172. If BC is produced to C′, CA to A′, AB to B′, where $CC' = AA' = BB' = x$, and if $\triangle A'B'C' = 2 \triangle ABC$, prove that

$$(a + b + c)x^2 + (bc + ca + ab)x = abc.$$

173. If, in a triangle ABC, R_1, R_2, R_3 are the radii of the circumcircles of the triangles IBC, ICA, IAB respectively, show that

(i) $R_1R_2R_3 = 2R^2r$,

(ii) $R_1^2 + R_2^2 + R_3^2 = 2R(2R - r)$.

CHAPTER X

SOLUTION OF TRIANGLES

§ 1. Introductory

THE process of calculating, from adequate data, the lengths of the sides and the magnitudes of the angles of a triangle is known as solving the triangle. This chapter will be devoted to a consideration of the various cases which arise, and of systematic methods of carrying out the necessary calculations.

From the geometrical theorems regarding congruence of triangles, it follows that a triangle is completely determined in the following cases:

(i) when one of the angles is known to be a right angle, and two of the other elements are given, one of these at least being a side;

(ii) when the three sides are given;

(iii) when two sides and the angle formed by them are given;

(iv) when one side and two angles are given.

In each of these cases the remaining elements of the triangle may be calculated by means of the formulæ established in Chapter IX.

In a fifth case,

(v) when two sides and the angle opposite to one of them are given,

it may, or may not, according to the size of the given angle and the relative lengths of the given sides, be possible to find the other elements of the triangle without ambiguity.

These five cases will be considered in §§ 3-7.

§ 2. Arrangement of Calculations

Calculations by means of logarithms should be arranged in tabular schemes, in order that the subsidiary arithmetical operations which are involved may be carried out with the greatest possible convenience, and that the danger of a slip occurring in one of these operations may be reduced to a minimum. Further, every opportunity should be taken to apply, at intermediate stages, tests of the accuracy of the results that have been obtained; so that, should an error be made, it may be detected and corrected as soon as possible, and so prevented from vitiating all the work that follows.

It is an advantage to draw up the entire scheme, or as much of it as possible, in blank before beginning the actual calculation. When this is done, the student is enabled to take a comprehensive view of the problem before him; and a saving of time and labour may be effected, as it will often be found possible to make several readings, required at different parts of the scheme, during a single reference to a particular table.

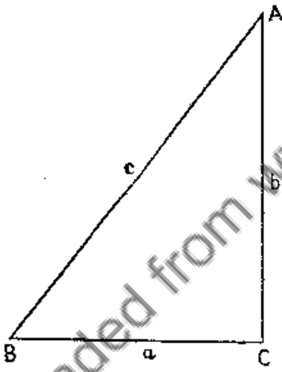

FIG. 1.

§ 3. One Angle Right

Suppose that it is known that, in the triangle ABC, $C = 90°$. If, in addition, two other elements are given, at least one of them being a side, the triangle may be solved by a variety of simple methods of which the following are illustrations:

Given a and b. Use the formulæ

$$\tan A = a/b, \text{ to find } A;$$
$$B = 90° - A, \text{ to find } B;$$
$$c = b \sec A, \text{ to find } c.$$

Given a and c.—Use the formulæ

$$\sin A = a/c, \text{ to find } A;$$
$$B = 90° - A, \text{ to find } B;$$
$$b = a \tan B, \text{ to find } b.$$

Given a and A.—Use the formulæ

$$B = 90^\circ - A, \text{ to find } B;$$
$$b = a \tan B, \text{ to find } b;$$
$$c = a \sec B, \text{ to find } c.$$

Given c and A. Use the formulæ

$$B = 90^\circ - A, \text{ to find } B;$$
$$a = c \sin A, \text{ to find } a;$$
$$b = c \sin B, \text{ to find } b.$$

Example.—Solve the triangle ABC, in which $C = 90^\circ$, $A = 38^\circ\ 26'$ and $c = 27{\cdot}35$.

We have at once $B = 90^\circ - A = 51^\circ\ 34'$. The rest of the calculation may be arranged as shown. Since $a = c \sin A$, $\log a = \log c + \log \sin A$; similarly, $\log b = \log c + \log \sin B$.

	Values.	Logarithms.
c	27·35	1·43695
sin A	sin 38° 26′	$\bar{1}$·79351
sin B	sin 51° 34′	$\bar{1}$·89394
a	17·000	1·23046
b	21·424	1·33089

$$\text{Results} \begin{cases} B = 51^\circ\ 34' \\ a = 17{\cdot}00 \\ b = 21{\cdot}42. \end{cases}$$

Note.—In filling up the above scheme, the following procedure should be observed. The first three entries in the second column should be made before any reading from the tables is taken. Next log c should be read off and entered in the third column, followed by log sin A and log sin B, which, added in turn to log c, give log a and log b. The table of antilogarithms then gives a and b for entry in the second column.

§ 4. Three Sides given

To solve the triangle ABC *when a, b and c are given.*

The method which has been found in practice to be the most convenient for this case, employs the formulæ

$$\Delta^2 = s(s-a)(s-b)(s-c), \qquad (1)$$
$$r = \Delta/s, \qquad (2)$$
$$\tan \tfrac{1}{2}A = r/(s-a); \ \tan \tfrac{1}{2}B = r/(s-b); \ \tan \tfrac{1}{2}C = r/(s-c). \qquad (3)$$

The corresponding scheme is shown in blank, with columns numbered for convenience of reference:

1		2		3		4		5	
	Values.		Values.		Logarithms.		Logarithms.		Values.
				Δ					
		s		s					
				r					
a		$s-a$		$s-a$		tan $\frac{1}{2}$A		$\frac{1}{2}$A	
b		$s-b$		$s-b$		tan $\frac{1}{2}$B		$\frac{1}{2}$B	
c		$s-c$		$s-c$		tan $\frac{1}{2}$C		$\frac{1}{2}$C	
$2s$		Check Sum $=2s$		Δ^2				Check Sum $=90°$	

The procedure in filling in the scheme for a particular triangle is then as follows.

In column 1 enter the given values of a, b and c; and add to obtain $2s$. This provides s, from which, by subtracting a, b, c in turn, the remaining entries in column 2 are found.

As a test of the accuracy of these additions and subtractions, add the four entries in column 2. The sum should be $4s-(a+b+c)$, that is, $2s$.

In column 3, enter the logarithms of s, $s-a$, $s-b$, $s-c$. Their sum is, by (1), log Δ^2; which is entered at the foot of the column, and divided by 2 to give log Δ for the head of the column. Subtraction of log s from log Δ gives, by (2), log r, which completes column 3.

Subtract, in turn, log $(s-a)$, log $(s-b)$, log $(s-c)$ from log r. The remainders, which, by (3), are respectively log tan $\frac{1}{2}$A, log tan $\frac{1}{2}$B, log tan $\frac{1}{2}$C, are entered in column 4.

Reference to the table of logarithms of tangents now gives the values of $\frac{1}{2}$A, $\frac{1}{2}$B, $\frac{1}{2}$C for column 5. Their sum should be 90°. In order that the values of A, B, C may be correct to the nearest minute, these half-angles should be read, when possible, correct to the nearest half-minute.

The values of Δ and r, if required, can be read from the table of antilogarithms, and shown in column 5.

The following example shows the scheme completed for a particular triangle. The student should prepare the scheme in blank, and fill it in independently.

Example.—In a triangle ABC,

$a = 20{\cdot}57$ inches, $b = 31{\cdot}69$ inches, $c = 24{\cdot}26$ inches.

Calculate the values of A, B, C, Δ, r.

	Values.		Values.		Logarithms.		Logarithms.		Values.
				Δ	2·39708			Δ	249·51
		s	38·26	s	1·58274				
				r	0·81434				6·5214
a	20·57	$s-a$	17·69	$s-a$	1·24771	tan ½A	$\bar{1}$·56663	½A	20° 14¼′
b	31·69	$s-b$	6·57	$s-b$	0·81757	tan ½B	$\bar{1}$·99677	½B	44° 47¼′
c	24·26	$s-c$	14·00	$s-c$	1·14613	tan ½C	$\bar{1}$·66821	½C	24° 58½′
$2s$	76·52	Check Sum $=2s$	76·52	Δ^2	4·79415			Check Sum $=90°$	90°

Results $\begin{cases} \text{A} = 40° \, 28\tfrac{1}{2}' ; \ \text{B} = 89° \, 34\tfrac{1}{2}' ; \ \text{C} = 49° \, 57' ; \\ \Delta = 249{\cdot}5 \text{ square inches} ; \ r = 6{\cdot}521 \text{ inches.} \end{cases}$

§ 5. Two Sides and the Included Angle given

To solve the triangle ABC, *when, for example, a, b and* C *are given.*

In this case we use the formulæ

$$\tan \tfrac{1}{2}(\text{A} - \text{B}) = \frac{a - b}{a + b} \tan \tfrac{1}{2}(\text{A} + \text{B}), \qquad (4)$$

$$\frac{a}{\sin \text{A}} = 2\text{R} = \frac{b}{\sin \text{B}}, \qquad (5)$$

$$c = 2\text{R} \sin \text{C}. \qquad (6)$$

Formula (4) is quoted in the form suitable when $a > b$. If $a < b$, the equivalent form obtained by interchanging a and A with b and B would be used.

The blank scheme is as follows:

	1 Values.	2 Logarithms.
$a-b$ $\tan \frac{1}{2}(A+B)$ $a+b$ $\tan \frac{1}{2}(A-B)$		
A B		
a $\sin A$ 2R $\sin C$ c		

	Values.	Logarithms.
b $\sin B$ 2R		
	Check	

In filling in the scheme for a particular triangle, first enter in the appropriate places the values of a, b, $(a - b)$ and $(a + b)$, and their logarithms.

Next find the value of $\frac{1}{2}(A + B)$, which is $90° - \frac{1}{2}C$, and enter it in column 1. Read $\log \tan \frac{1}{2}(A + B)$ from the tables, add it to $\log (a - b)$; and from the sum subtract $\log (a + b)$. The result is, by (4), $\log \tan \frac{1}{2}(A - B)$; and $\frac{1}{2}(A - B)$ may then be found from the table of logarithms of tangents.

Addition of $\frac{1}{2}(A - B)$ to $\frac{1}{2}(A + B)$ gives A; subtraction of $\frac{1}{2}(A - B)$ from $\frac{1}{2}(A + B)$ gives B.

One reference to the table of logarithms of sines now gives the logarithms of sin A, sin B and sin C.

$$\text{By (5), } \log 2R = \log a - \log \sin A$$
$$= \log b - \log \sin B;$$

and log 2R should be found from each of these expressions, one of the calculations being annexed to the scheme as a check on the other.

Finally, by (6), log c is found as log 2R + log sin C; and c is read from the table of antilogarithms, from which 2R may also be found if necessary.

An example of a completed scheme will now be given.

Example.—In a triangle ABC,

$$a = 219{\cdot}7 \text{ feet}, \quad b = 109{\cdot}3 \text{ feet}, \quad C = 45^\circ\ 33'.$$

Calculate A, B, c, R.

	Values.	Logarithms.
$a-b$	110·4	2·04293
$\tan \frac{1}{2}(A+B)$	tan 67° $13\frac{1}{2}'$	0·37691
—		2·41984
$a+b$	329·0	2·51720
$\tan \frac{1}{2}(A-B)$	tan 38° 38′	$\bar{1}\cdot 90264$
A	105° $51\frac{1}{2}'$	
B	28° $35\frac{1}{2}'$	
a	219·7	2·34184
sin A		$\bar{1}\cdot 98315$
2R	228·39	2·35869
sin C		$\bar{1}\cdot 85361$
c	163·04	2·21230

	Values.	Logarithms.
b	109·3	2·03864
sin B		$\bar{1}\cdot 67994$
2R		2·35870
		Check

Results $\begin{cases} A = 105^\circ\ 51\frac{1}{2}'; & B = 28^\circ\ 35\frac{1}{2}'; \\ c = 163{\cdot}0 \text{ feet}; & R = 114{\cdot}2 \text{ feet.} \end{cases}$

§ 6. One Side and two Angles given

To solve the triangle ABC *when, for example,* a, A, B *are given.*

The third angle C is found directly as the supplement of (A + B); and for the rest of the solution the Law of Sines is the only formula that is needed.

Log 2R is found as log a − log sin A, since 2R = a/sin A.

Then log b and log c are obtained by adding in turn log sin B and log sin C to log 2R.

The tabular scheme for this case takes the simple form shown in the following example:

Example.—In the triangle ABC, $a = 262$, $A = 45° 13'$ and $B = 99° 27'$. Calculate C, b, c, R.

	Values.	Logarithms.
a	262	2·41830
sin A	sin 45° 13′	$\bar{1}$·85112
2R	369·13	2·56718
sin B	sin 99° 27′	$\bar{1}$·99406
sin C	sin 35° 20′	$\bar{1}$·76217
b	364·12	2·56124
c	213·47	2·32935

$$\text{Results} \begin{cases} C = 35° 20'; & b = 364{\cdot}1; \\ c = 213{\cdot}5; & R = 184{\cdot}6. \end{cases}$$

§ 7. Two Sides and the Angle opposite one of them given

To solve the triangle ABC, *in which, for example, a, b and* A *are given.*

As in the case discussed in § 6, the Law of Sines is the only triangle formula that is needed. We may proceed thus:

(i) find log 2R as log a − log sin A;
(ii) find B from the formula $\sin B = b/2R$;
(iii) find C as the supplement of $(A + B)$, and c from the formula $c = 2R \sin C$.

If we assume that the data are consistent, that is, that they refer to a real triangle, then $b \leq 2R$, since no side of the triangle can be greater than the diameter of the circum-circle. Hence at stage (ii) the following possibilities arise:

(α) $\dfrac{b}{2R} = 1$, in which case B is definitely a right angle;

(β) $\dfrac{b}{2\text{R}} < 1$, in which case the equation $\sin \text{B} = \dfrac{b}{2\text{R}}$ leads to two possible values of B, one acute, the other obtuse. But, if $a \geqq b$, then $\text{A} \geqq \text{B}$, and the angle B must be acute, since there cannot be two obtuse angles in a triangle.

The Ambiguous Case.—If, on the other hand, $a < b$, so that $\text{A} < \text{B}$ and the given angle A is bound to be acute, both of the values of B given by (ii) are valid: B may be acute and greater than A, or obtuse.

Hence, if the given angle is opposite the smaller of the given sides, there may be ambiguity. Unless, as in (α), the angle opposite the larger of the given sides is right, two different triangles can be found to fit the data, and further information is needed if the triangle is to be determined uniquely.

The following example shows how the solution may be arranged in the case where ambiguity occurs:

Example.—Solve the triangle ABC in which

$$a = 60{\cdot}21, \quad b = 107{\cdot}2, \quad \text{A} = 33° \, 16'.$$

	Values.	Logarithms.	OR Values.	OR Logarithms.
b	107·2	2·03019		
a	60·21	1·77967		
sin A	sin 33° 16′	$\bar{1}$·73919		
2R		2·04048		
sinB	sin 77° 35′	$\bar{1}$·98971	sin 102° 25′	
sin C	sin 69° 9′	$\bar{1}$·97058	sin 44° 19′	$\bar{1}$·84424
c	102·58	2·01106	76·689	1·88472

Results $\begin{cases} \textit{Either } \text{B} = 77° \, 35'; \ \text{C} = 69° \, 9'; \ c = 102{\cdot}6; \\ \textit{or } \text{B} = 102° \, 25'; \ \text{C} = 44° \, 19'; \ c = 76{\cdot}69. \end{cases}$

It is instructive to examine the above ambiguous case geometrically.

Let $\angle$DAC be equal to the given angle A, and let AC be equal to the given side b. Then the third vertex B of the required triangle ABC must lie on AD, and on the same side of A as D. To find its position, take C as centre, the given side a as radius, and describe a circle. If the data are consistent, this circle will meet AD. It will *either* (Fig. 2)

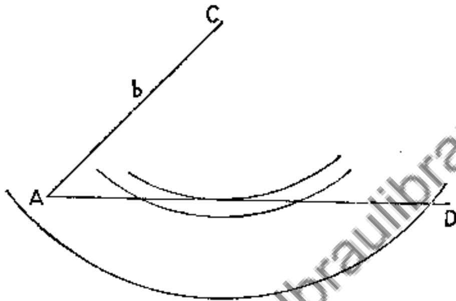

FIG. 2.

(i) *touch* AD, at B, the third vertex of the triangle ABC;

or (ii) cut the straight line AD in two distinct points, B (on the same side of A as D), and B_1 (on the opposite side of A), B being the required third vertex;

or (iii) cut the straight line AD in two distinct points, B and B_1, which are both on the same side of A as D. This can happen only when $a < b$, and of course, as that condition implies, the angle A is acute. Each of the triangles ABC, AB_1C is consistent with the data.

EXAMPLES X

Solve the right-angled triangle ABC in the following cases:

1. $C = 90°$, $a = 43$, $b = 71$.
   Ans. $A = 31° 12'$, $B = 58° 48'$, $c = 83{\cdot}01$.
2. $A = 90°$, $b = 2{\cdot}735$, $c = 1{\cdot}989$.
   Ans. $B = 53° 58\frac{1}{2}'$, $C = 36° 1\frac{1}{2}'$, $a = 3{\cdot}382$.
3. $B = 90°$, $c = 647{\cdot}2$, $a = 125{\cdot}8$.
   Ans. $C = 79° 0'$, $A = 11° 0'$, $b = 659{\cdot}3$.
4. $C = 90°$, $a = 543$, $c = 946$.
   Ans. $A = 35° 2'$, $B = 54° 58'$, $b = 774{\cdot}6$.
5. $A = 90°$, $a = 9{\cdot}343$, $b = 8{\cdot}011$.
   Ans. $B = 59° 1\frac{1}{2}'$, $C = 30° 58\frac{1}{2}'$, $c = 4{\cdot}808$.

6. $C = 90°$, $B = 37°$, $b = 58$.
Ans. $A = 53°$, $a = 76{\cdot}97$, $c = 96{\cdot}38$.

7. $B = 90°$, $C = 63°$, $a = 492{\cdot}5$.
Ans. $A = 27°$, $c = 966{\cdot}6$, $b = 1085$.

8. $A = 90°$, $B = 78°\ 13'$, $a = 1489$.
Ans. $C = 11°\ 47'$, $b = 1458$, $c = 304{\cdot}1$.

Solve the triangle ABC in the following cases:

9. $a = 25$, $b = 37$, $c = 48$.
Ans. $A = 30°\ 54'$, $B = 49°\ 27'$, $C = 99°\ 39'$.

10. $a = 69$, $b = 38$, $c = 57$.
Ans. $A = 90°\ 54'$, $B = 33°\ 25'$, $C = 55°\ 41'$.

11. $a = 725$, $b = 548$, $c = 474$.
Ans. $A = 90°\ 4'$, $B = 49°\ 6'$, $C = 40°\ 50'$.

12. $a = 78$, $b = 148$, $c = 218$.
Ans. $A = 11°\ 0'$; $B = 21°\ 13'$, $C = 147°\ 48'$.

13. $a = 10{\cdot}58$, $b = 31{\cdot}69$, $c = 24{\cdot}26$.
Ans. $A = 15°\ 37'$, $B = 126°\ 17'$, $C = 38°\ 6'$.

14. $a = 96{\cdot}13$, $b = 62{\cdot}42$, $c = 75{\cdot}05$.
Ans. $A = 88°\ 14'$, $B = 40°\ 28'$, $C = 51°\ 18'$.

15. $a = 98{\cdot}5$, $b = 175{\cdot}3$, $c = 221{\cdot}4$.
Ans. $A = 25°\ 32'$, $B = 50°\ 5'$, $C = 104°\ 24'$.

16. $a = 162{\cdot}4$, $b = 219{\cdot}5$, $c = 109{\cdot}3$.
Ans. $A = 45°\ 18'$, $B = 106°\ 8'$, $C = 28°\ 35'$.

17. $a = 883$, $b = 729$, $c = 1014$.
Ans. $A = 58°\ 10'$, $B = 44°\ 32'$, $C = 77°\ 18'$.

18. $a = 105$, $b = 130$, $c = 155$.
Ans. $A = 42°\ 6'$, $B = 56°\ 7'$, $C = 81°\ 47'$.

19. $a = 312$, $b = 404$, $c = 116$.
Ans. $A = 32°\ 11'$, $B = 136°\ 24'$, $C = 11°\ 25'$.

20. $a = 49{\cdot}2$, $b = 86{\cdot}1$, $c = 110{\cdot}7$.
Ans. $A = 25°\ 13'$, $B = 48°\ 11'$, $C = 106°\ 36'$.

21. $a = 7142$, $b = 5387$, $c = 6029$.
Ans. $A = 77°\ 14'$, $B = 47°\ 21'$, $C = 55°\ 25'$.

22. $a = 11{\cdot}1$, $b = 19{\cdot}5$, $c = 27{\cdot}9$.
Ans. $A = 17°\ 54'$, $B = 32°\ 40'$, $C = 129°\ 26'$.

23. $a = 17{\cdot}39$, $b = 22{\cdot}88$, $c = 15{\cdot}47$.
Ans. $A = 49°\ 26'$, $B = 88°\ 3'$, $C = 42°\ 31'$.

24. $a = 5134$, $b = 7268$, $c = 9313$.
Ans. $A = 33°\ 16'$, $B = 50°\ 56'$, $C = 95°\ 48'$.

25. $a = 120$, $b = 29$, $c = 101$.
Ans. $A = 124°\ 59'$, $B = 11°\ 25'$, $C = 43°\ 36'$.

26. $a = 61, \quad b = 229, \quad c = 232.$
Ans. $A = 15° 11', \quad B = 79° 37', \quad C = 85° 12'.$
27. $a = 41{\cdot}39, \quad b = 55{\cdot}28, \quad c = 69{\cdot}17.$
Ans. $A = 36° 45', \quad B = 53° 3', \quad C = 90° 12'.$
28. $a = 37, \quad b = 44, \quad c = 15.$
Ans. $A = 53° 8', \quad B = 107° 57', \quad C = 18° 55'.$
29. $b = 63{\cdot}2, \quad c = 41{\cdot}0, \quad A = 50° 16'.$
Ans. $B = 89° 17\frac{1}{2}', \quad C = 40° 26\frac{1}{2}', \quad a = 48{\cdot}61.$
30. $b = 330{\cdot}9, \quad c = 551{\cdot}4, \quad A = 30° 24'.$
Ans. $B = 32° 11\frac{1}{2}', \quad C = 117° 24\frac{1}{2}', \quad a = 314{\cdot}3.$
31. $a = 2197, \quad b = 1093, \quad C = 45° 33'.$
Ans. $A = 105° 51\frac{1}{2}', \quad B = 28° 35\frac{1}{2}', \quad c = 1630.$
32. $a = 1010, \quad b = 1892, \quad C = 41° 1'.$
Ans. $A = 30° 24', \quad B = 108° 35', \quad c = 1310.$
33. $c = 830, \quad a = 1734, \quad B = 118° 5'.$
Ans. $C = 19° 1', \quad A = 42° 54', \quad b = 2247.$
34. $c = 1820, \quad a = 1264, \quad B = 70° 31'.$
Ans. $C = 69° 3', \quad A = 40° 26', \quad b = 1837.$
35. $b = 41{\cdot}67, \quad c = 82{\cdot}32, \quad A = 71° 14'.$
Ans. $B = 29° 47\frac{1}{2}', \quad C = 78° 58\frac{1}{2}', \quad a = 79{\cdot}41.$
36. $b = 1321, \quad c = 808{\cdot}6, \quad A = 127° 30'.$
Ans. $B = 33° 1', \quad C = 19° 29', \quad a = 1923.$
37. $b = 372{\cdot}4, \quad c = 193{\cdot}9, \quad A = 68°.$
Ans. $B = 81° 3', \quad C = 30° 57', \quad a = 349{\cdot}6.$
38. $a = 723{\cdot}5, \quad b = 468{\cdot}5, \quad C = 145° 17'.$
Ans. $A = 21° 11', \quad B = 13° 32', \quad c = 1140.$
39. $a = 121, \quad b = 393, \quad C = 67° 15'.$
Ans. $A = 17° 52', \quad B = 94° 53', \quad c = 363{\cdot}7.$
40. $a = 3975, \quad b = 3320, \quad C = 44° 41'.$
Ans. $A = 79° 59', \quad B = 55° 20', \quad c = 2838.$
41. $c = 1{\cdot}656, \quad a = 3{\cdot}724, \quad B = 85° 58'.$
Ans. $C = 24° 36', \quad A = 69° 26', \quad b = 3{\cdot}968.$
42. $c = 45{\cdot}43, \quad a = 42{\cdot}68, \quad B = 93° 47'.$
Ans. $C = 44° 47', \quad A = 41° 26', \quad b = 64{\cdot}35.$
43. $c = 5175, \quad a = 6994, \quad B = 12° 22'.$
Ans. $C = 29° 45', \quad A = 137° 53', \quad b = 2234.$
44. $b = 60{\cdot}21, \quad c = 25{\cdot}09, \quad A = 89° 51'.$
Ans. $B = 67° 30\frac{1}{2}', \quad C = 22° 38\frac{1}{2}', \quad a = 65{\cdot}17.$
45. $b = 250, \quad c = 1026, \quad A = 123° 12'.$
Ans. $B = 10° 12', \quad C = 46° 36', \quad a = 1181.$

46. $a = 275{\cdot}5$, $b = 342{\cdot}4$, $C = 137° 13'$.
Ans. $A = 18° 58'$, $B = 23° 49'$, $c = 575{\cdot}8$.

47. $c = 989{\cdot}5$, $a = 893{\cdot}7$, $B = 18° 36'$.
Ans. $C = 97° 57\frac{1}{2}'$, $A = 63° 26\frac{1}{2}'$, $b = 318{\cdot}7$.

48. $c = 49{\cdot}98$, $a = 52{\cdot}52$. $B = 100° 50'$.
Ans. $C = 38° 24\frac{1}{2}'$, $A = 40° 45\frac{1}{2}'$, $b = 79{\cdot}01$.

49. $a = 3{\cdot}606$, $B = 47° 18'$, $C = 62° 45'$.
Ans. $A = 69° 57'$, $b = 2{\cdot}821$, $c = 3{\cdot}413$.

50. $c = 1571$, $B = 52° 6'$, $C = 109° 20'$.
Ans. $A = 18° 34'$, $b = 1314$, $a = 530{\cdot}1$.

51. $b = 107{\cdot}2$, $A = 33° 16'$, $C = 44° 19'$.
Ans. $B = 102° 25'$, $a = 60{\cdot}21$, $c = 76{\cdot}69$.

52. $a = 262$, $A = 45° 13'$, $B = 99° 27'$.
Ans. $C = 35° 20'$, $c = 213{\cdot}5$, $b = 364{\cdot}1$.

53. $c = 320{\cdot}6$, $A = 52° 9'$, $B = 56° 56'$.
Ans. $C = 70° 55'$, $a = 267{\cdot}9$, $b = 284{\cdot}3$.

54. $b = 35{\cdot}25$, $A = 63° 27'$, $C = 71° 52'$.
Ans. $B = 44° 41'$, $a = 44{\cdot}84$, $c = 47{\cdot}64$.

55. $a = 6182$, $b = 8574$, $B = 57° 12'$.
Ans. $A = 37° 18'$, $C = 85° 30'$, $c = 10170$.

56. $c = 47{\cdot}23$, $a = 56{\cdot}55$, $C = 48° 37'$.
Ans. $A = 63° 57'$, $B = 67° 26'$, $b = 58{\cdot}13$;
or $A = 116° 3'$, $B = 15° 20'$, $b = 16{\cdot}64$.

57. $c = 946$, $a = 543$, $A = 34° 45'$.
Ans. $C = 83° 14'$, $B = 62° 1'$, $b = 841{\cdot}3$;
or $C = 96° 46'$, $B = 48° 29'$, $b = 713{\cdot}3$.

58. $b = 360{\cdot}7$, $c = 384{\cdot}2$, $B = 32° 19'$.
Ans. $C = 34° 43'$, $A = 112° 58'$, $a = 621{\cdot}2$;
or $C = 145° 17'$, $A = 2° 24'$, $a = 28{\cdot}26$.

59. A bears W. 62° 19′ N. at C, and B bears E. 43° 38′ N. at C. If AC = 56·3 and BC = 77·5, calculate the length of AB and the bearing of B at A.
Ans. AB = 82·3, E. 2° $31\frac{1}{2}'$ N.

60. Solve the triangle ABC in which $c = 205$, $a = 316$ and $A - C = 48° 51'$.
Ans. $A = 89° 17\frac{1}{2}'$, $B = 50° 16'$, $C = 40° 26\frac{1}{2}'$, $b = 243{\cdot}0$.

61. Solve the triangle ABC in which CA = 22·31, AB = 57·50 and the angles opposite these sides differ by 82° 40′.
Ans. $A = 53° 15'$, $B = 22° 2\frac{1}{2}'$, $C = 104° 42\frac{1}{2}'$, $a = 47{\cdot}63$.

62. In a triangle ABC, $a = 297$, $b = 215$ and $A - B = 74° 10'$. Find A, B, C, c and R.

Ans. $A = 115° 7'$ $B = 40° 57'$, $C = 23° 56'$, $c = 133$
$R = 164$.

63. Show that, in any triangle ABC,

$$(b - c) \cos \tfrac{1}{2}A = a \sin \tfrac{1}{2}(B - C),$$

and solve the triangle when $b - c = 48$, $B = 67°$, $C = 56°$.

Ans. $a = 440{\cdot}1$, $b = 483{\cdot}1$, $c = 435{\cdot}1$, $A = 57°$.

64. Solve the triangle ABC in which $a = 43{\cdot}21$, $b - c = 11{\cdot}12$ and $A = 52° 38'$.

Ans. $B = 77° 1'$, $C = 50° 21'$, $b = 52{\cdot}98$, $c = 41{\cdot}86$.

65. Find all the angles of the triangle ABC in which the circum-radius is 7 inches, a is 12 inches and $b - c$ is 5 inches.

Ans. $A = 59°$, $B = 81° 46'$, $C = 39° 14'$.

66. Solve the triangle ABC in which

$$c = 21{\cdot}83, \quad a - b = 11{\cdot}95, \quad A - B = 45° 21'.$$

Ans. $A = 67° 26\tfrac{1}{2}'$, $B = 22° 5\tfrac{1}{2}'$, $C = 90° 28'$, $a = 20{\cdot}16$,
$b = 8{\cdot}21$.

67. Show that, in any triangle ABC,

$$a \cos \tfrac{1}{2}(B - C) = (b + c) \sin \tfrac{1}{2}A,$$

and solve the triangle in which $a = 337$, $b + c = 489$, $A = 64°$.

Ans. $B = 97° 44\tfrac{1}{2}'$, $C = 18° 15\tfrac{1}{2}'$, $b = 371{\cdot}5$, $c = 117{\cdot}5$.

68. If, in the triangle ABC, $b + c = 6935$, $a = 3758$, $A = 38° 40'$, find B and C.

Ans. $B = 123° 1'$, $C = 18° 19'$.

69. Solve the triangle ABC in which $a + b + c = 354$ inches, $a = 99$ inches, $R = 102$ inches.

Ans. $A = 29° 2'$, $B = 125° 16'$, $C = 25° 42'$, $b = 166{\cdot}6$ ins.,
$c = 88{\cdot}5$ ins.

70. Show that, in any triangle ABC,

$$(b + c) \sin \tfrac{1}{2}A = a \sin (\tfrac{1}{2}A + C),$$

and solve the triangle when $a = 5$, $b + c = 11$, $A = 50°$.

Ans. $B = 86° 36'$, $C = 43° 24'$, $b = 6{\cdot}516$, $c = 4{\cdot}485$.

71. Solve the triangle ABC for which

$$a = 37{\cdot}42, \quad b + c = 62{\cdot}51, \quad A = 72° 36'.$$

Ans. $B = 45° 11'$, $C = 62° 13'$, $b = 27{\cdot}82$, $c = 34{\cdot}69$.

72. Solve the triangle ABC for which

$$a = 62{\cdot}81, \quad b + c = 83{\cdot}24, \quad A = 65° 32'.$$

Ans. $B = 13° 4'$, $C = 101° 24'$, $b = 15{\cdot}60$, $c = 67{\cdot}65$.

73. In any triangle ABC show that

$$(a + b + c) \tan \tfrac{1}{2}B = (a + b - c) \cot \tfrac{1}{2}A,$$

and solve the triangle in which

$$a + b = 95, \quad c = 60, \quad A = 55°.$$

Ans. $B = 46° \, 54'$, $C = 78° \, 6'$, $a = 50{\cdot}23$, $b = 44{\cdot}77$.

74. Solve the triangle ABC in which

$$a + b = 135, \quad c = 77, \quad A = 41° \, 12'.$$

Ans. $B = 72° \, 6'$, $C = 66° \, 42'$, $a = 55{\cdot}22$, $b = 79{\cdot}78$.

75. Solve the triangle ABC in which

$$a + b = 143{\cdot}2, \quad c = 75{\cdot}5, \quad A = 50° \, 26'.$$

Ans. $B = 66° \, 38'$, $C = 62° \, 56'$, $a = 65{\cdot}36$, $b = 77{\cdot}83$.

76. Find the lengths of the sides of the triangle ABC in which $A = 53° \, 14'$, $B = 67° \, 22'$, and $a + b + c = 74{\cdot}18$ inches.

Ans. $a = 22{\cdot}99$ ins., $b = 26{\cdot}49$ ins., $c = 24{\cdot}70$ ins.

77. Solve the triangle ABC in which

$$A = 37° \, 21', \quad B = 84° \, 33', \quad s = 28{\cdot}64.$$

Ans. $C = 58° \, 6'$, $a = 14{\cdot}18$, $b = 23{\cdot}26$, $c = 19{\cdot}84$.

78. Show that, in any triangle ABC, $a = (b + c) \cos \theta$, where $0 < \theta < 90°$ and $(b + c) \sin \theta = 2\sqrt{(bc)} \cos \tfrac{1}{2}A$. If $b = 15{\cdot}58$, $c = 12{\cdot}24$, $A = 62° \, 44'$, find θ and a.

Ans. $\theta = 57° \, 57\tfrac{1}{2}'$, $a = 14{\cdot}76$.

79. If $b = 120$, $c = 230$, $A = 128° \, 36'$, find a, B and C.

Ans. $a = 319{\cdot}0$, $B = 17° \, 6'$, $C = 34° \, 18'$.

80. Show that, in any triangle ABC, $a = (b - c) \sec \phi$, where $0 < \phi < 180°$ and $(b - c) \tan \phi = 2\sqrt{(bc)} \, . \sin \tfrac{1}{2}A$. If $b = 27{\cdot}63$, $c = 15{\cdot}25$, $A = 105° \, 18'$, find a, B, C.

Ans. $a = 34{\cdot}90$, $B = 49° \, 47'$, $C = 24° \, 55'$.

81. Solve the triangle ABC in which $A = 82°$,

$$\cos B - \cos C = 0{\cdot}25$$

and $BC = 12{\cdot}4$ feet.

Ans. $b = 7{\cdot}96$ ft., $c = 10{\cdot}68$ ft., $B = 39° \, 28'$, $C = 58° \, 32'$.

82. Show that in any triangle ABC

$$s = R(\sin A + \sin B + \sin C),$$

and solve the triangle in which $B = 61° \, 40'$, $C = 44° \, 27'$, $a + b + c = 84{\cdot}30$.

Ans. $A = 73° \, 53'$, $a = 31{\cdot}87$, $b = 29{\cdot}20$, $c = 23{\cdot}23$.

83. If, in the triangle ABC, $k = \cos A/\cos B$, prove that

$$\tan^2 \tfrac{1}{2}(A + B) = \frac{a+b}{a-b} \cdot \frac{1-k}{1+k},$$

and hence solve the triangle for which $a = 20$, $b = 15$, $2\cos A = \cos B$.

Ans. $c = 19{\cdot}62$, $A = 69°\ 6'$, $B = 44°\ 29'$, $C = 66°\ 25'$.

84. Show that the area of a triangle ABC is equal to

$$\tfrac{1}{2}a^2 \sin B \sin C \operatorname{cosec} A,$$

and hence find the lengths of the sides of the triangle in which $A = 62°$, $B = 43°$, and the area is 540 square units.

Ans. $a = 38{\cdot}05$, $b = 29{\cdot}39$, $c = 41{\cdot}62$.

85. If the internal bisector of the angle A of a triangle ABC meets the opposite side in D, show that

$$AD = \frac{bc}{2R\cos\tfrac{1}{2}(B - C)}.$$

If $A = 67°$, $B = 48°$, $R = 125$, find the length of AD.

Ans. $170{\cdot}2$.

CHAPTER XI

HEIGHTS AND DISTANCES

§ 1. Introductory

In this chapter a number of problems will be considered in which the object is to calculate, from data consisting of distances and angles which can be measured directly, distances and angles which cannot be measured directly. Such problems arise in the work of the surveyor, in which elementary trigonometry finds some of its most important practical applications. For a description of the instruments and the methods employed by the surveyor in measuring angles and distances, reference should be made to any textbook on surveying.

§ 2. Angles of Elevation and Depression

Let an object C (Fig. 1) be observed from a point A, and let D be the projection of C on the horizontal plane through

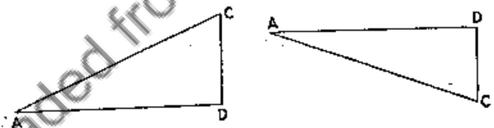

Fig. 1.

A. Then the angtle DAC, which the straight line from the point of obse rvaion to the object observed makes with the horizontal, is called *the angle of elevation of* C *from* A or *the angle of depression of* C *from* A, according as C is above or below the horizontal plane through A. The terms *elevation* and *depression* are often used instead of angle of elevation and angle of depression.

Since DC = AD tan DAC, the distance DC can be calculated if AD and $\angle$DAC can be measured.

§ 3. To Determine the Height of a Distant Object

For the present the effect of the curvature of the earth will be neglected, and the terms *on the same level as* A and *in the horizontal plane through* A will be regarded as synonymous.

Let C be an inaccessible object whose height h above the horizontal plane through a point of observation A is to be found.

It is necessary to use a second point of observation B, which should if possible be taken in the horizontal plane through A, and so situated that its distance a from A can be measured. AB is called *the base line*. The unknown distance h may then be expressed in terms of the known

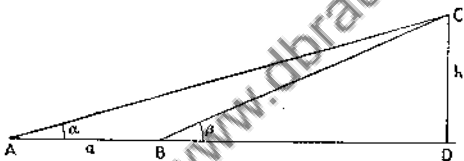

FIG. 2.

distance a, and circular functions of angles which can be measured by theodolite or sextant.

In what follows it will be understood that the base line is horizontal, unless the contrary is stated.

CASE 1.—*When the base line is in a vertical plane through* C. (Fig. 2.)

In this case the only angular observations which are needed are the angles of elevation α and β of C from A and B respectively. When these are known, h may be connected with a by means of the Law of Sines applied to the triangle ABC, it being noted that, from the right-angled triangle BDC, $BC = h/\sin\beta$.

Thus $$\frac{BC}{\sin BAC} = \frac{AB}{\sin ACB},$$

whence $$\frac{h/\sin\beta}{\sin\alpha} = \frac{a}{\sin(\beta - \alpha)}.$$

Therefore $$h = \frac{a \sin \alpha \sin \beta}{\sin (\beta - \alpha)} \quad . \quad . \quad . \quad (1)$$

If h is to be evaluated by means of logarithms, (1) should for greater convenience be written

$$h = a \sin \alpha \sin \beta \operatorname{cosec} (\beta - \alpha).$$

CASE 2.—*When the base line is at right angles to* AD. (Fig. 3.)

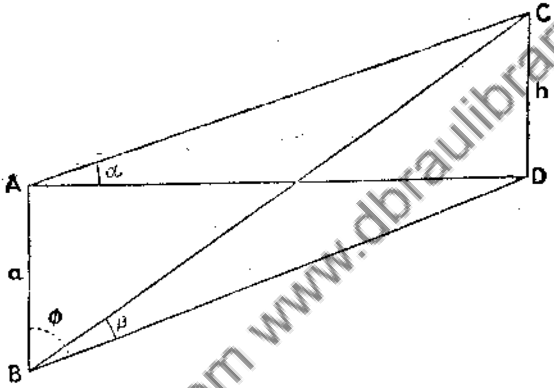

FIG. 3.

Two angular observations are required in this case also.

(i) Let the elevations α and β be measured by theodolite. Since CD is perpendicular to the plane ABD,

$$AD = h \cot \alpha \quad \text{and} \quad BD = h \cot \beta.$$

Hence, from the right-angled triangle DAB,

$$h^2 \cot^2 \beta = a^2 + h^2 \cot^2 \alpha.$$

Thus $$h^2 = \frac{a^2}{\cot^2 \beta - \cot^2 \alpha}$$

$$= \frac{a^2 \sin^2 \alpha \sin^2 \beta}{\sin (\alpha + \beta) \sin (\alpha - \beta)}.$$

Therefore $$h = \frac{a \sin \alpha \sin \beta}{\sqrt{\{\sin (\alpha + \beta) \sin (\alpha - \beta)\}}} \quad . \quad . \quad (2)$$

The student should verify that the same result is obtained by using the right-angled triangle CAB, in which $AC = h/\sin\alpha$ and $BC = h/\sin\beta$. Alternatively, we might proceed thus.

(ii) Let the elevation β of C from B, and the angle DBA, which is in a horizontal plane, be measured by theodolite.

Then, if $\angle DBA = \phi$, we have, from the right-angled triangles DAB and BDC,

$$h = BD \tan\beta = a \sec\phi \tan\beta, \quad . \quad . \quad (3)$$

a formula more convenient in practice than (2).

Example.—If the base line AB is taken so that $\angle ADB = \frac{1}{2}\pi$, show that $h = a/\sqrt{(\cot^2\alpha + \cot^2\beta)}$.

CASE 3.—*When the base line is oblique to* AD. (Fig. 4.)

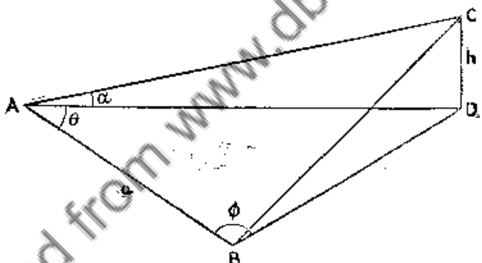

FIG. 4.

In this case three angular observations are needed.

(i) Let the elevation α of C from A, and the horizontal angles BAD (θ) and DBA (ϕ), be measured by theodolite.

Applying the Law of Sines to the triangle ABD, whose angles are known, we have

$$\frac{AD}{\sin\phi} = \frac{AB}{\sin(\theta + \phi)};$$

and, since $AD = h/\tan\alpha$, this gives

$$h = \frac{a \tan\alpha \sin\phi}{\sin(\theta + \phi)} \quad . \quad . \quad . \quad (4)$$

If the elevation β from B is used instead of α, the result is

$$h = \frac{a \tan \beta \sin \theta}{\sin (\theta + \phi)} \quad . \quad . \quad . \quad (5)$$

The student will note that (3) may be deduced from (5) by putting $\theta = 90°$.

Alternatively (Fig. 5):

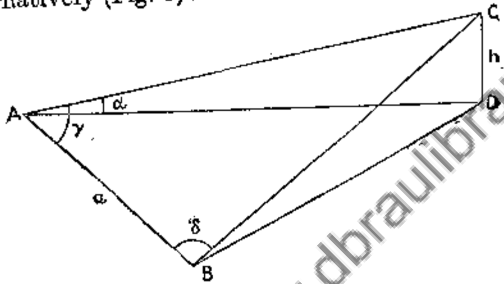

FIG. 5.

(ii) Let the elevation α be measured by theodolite, and the angles BAC (γ) and CBA (δ) by sextant.

Then, from the triangle ABC,

$$\frac{AC}{\sin \delta} = \frac{AB}{\sin (\gamma + \delta)};$$

and, since $AC = h/\sin \alpha$,

$$h = \frac{a \sin \alpha \sin \delta}{\sin (\gamma + \delta)}. \quad . \quad . \quad . \quad (6)$$

It should be noted that the result (6) holds whether the base line is horizontal or not, and is on that account more generally useful than (5).

If in (6) we replace γ by α, and δ by $180° - \beta$, we obtain (1).

§ 4. To Determine the Distance between two Inaccessible Objects

Let C and D (Fig. 6) be the objects whose distance d apart is to be found, and let a suitable base line AB of length a be chosen. As a rule AB and CD will not be coplanar.

If the angles BAC (α) and CBA (β) are measured, the length of BC can be found from the triangle ABC in which one side and two angles are known. Similarly, if the angles BAD (γ) and DBA (δ) are measured, the length of BD can be found from the triangle ABD. The Law of Sines gives

$$BC = a \sin \alpha / \sin (\alpha + \beta) ; \quad BD = a \sin \gamma / \sin (\gamma + \delta).$$

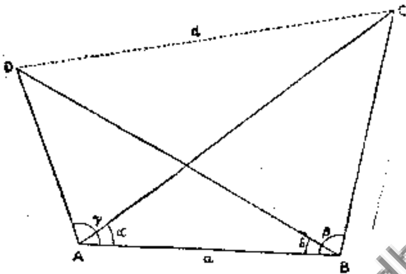

FIG. 6.

Now, in the triangle BCD, the lengths of the sides BC and BD are known; if, therefore, the angle CBD which they form is measured, the length of the third side CD can be calculated (Chap. X, § 5).

Of course, should AB and CD be coplanar, the angle CBD $= \beta - \delta$.

Alternatively, we might calculate AC and AD from the triangles ABC and ABD, measure the angle CAD, and calculate CD from the triangle ACD. If a careful check were considered desirable, CD might be evaluated by both methods.

Example.—A, B, C, D are four points which are not coplanar; AB = 1742 yards; BC and BD subtend at A angles of 61° 23′ and 110° 47′ respectively; AC, CD and DA subtend at B angles of 92° 12′, 65° 32′ and 31° 49′ respectively. Calculate, correct to the nearest yard, the length of CD.

Referring to Fig. 6, we have $a = 1742$, $\alpha = 61° \ 23'$, $\beta = 92° \ 12'$, $\gamma = 110° \ 47'$, $\delta = 31° \ 49'$ and $\angle CBD = 65° \ 32'$. Hence

$$BC = 1742 \sin 61° \ 23' \operatorname{cosec} 153° \ 35'$$
$$= 1742 \sin 61° \ 23' \operatorname{cosec} 26° \ 25' ;$$

and

$$BD = 1742 \sin 110° \ 47' \operatorname{cosec} 142° \ 36'$$
$$= 1742 \sin 69° \ 13' \operatorname{cosec} 37° \ 24'.$$

Having found BC (d) and BD (c), and knowing $\angle$CBD (B), we use the scheme of Chap. X, § 5, to calculate CD (b). The formulæ involved are:

$$\tan \tfrac{1}{2}(\mathrm{D} - \mathrm{C}) = \frac{d - c}{d + c} \tan \tfrac{1}{2}(\mathrm{D} + \mathrm{C});$$

$$d/\sin \mathrm{D} = 2\mathrm{R}\ (= c/\sin \mathrm{C})\ ;\ \ b = 2\mathrm{R} \sin \mathrm{B}.$$

The logarithmic work is set out below:

	Logarithms.		Logarithms.
1742 sin 61° 23′ cosec 26° 25′	3·24105 $\bar{1}$·94342 0·35175	1742 sin 69° 13′ cosec 37° 24′	3·24105 $\bar{1}$·97078 0·21654
BC = 3437·4	3·53622	BD = 2681·4	3·42837

	Values.	Logarithms.			
$d-c$ $\tan \frac{1}{2}(\mathrm{D}+\mathrm{C})$ — $d+c$ $\tan \frac{1}{2}(\mathrm{D}-\mathrm{C})$ D C	756·0 tan 57° 14′ — 6118·8 tan 10° 52′ 68° 6′ 46° 22′	2·87852 0·19137 3·06989 3·78667 $\bar{1}$·28322			
d sin D 2R sin B b	3437·4 sin 65° 32′ 3372·0	3·53622 $\bar{1}$·96747 3·56875 $\bar{1}$·95914 3·52789	c sin C 2R	2681·4	3·42837 $\bar{1}$·85960 3·56877

Result: CD = 3372 yards.

§ 5. Miscellaneous Problems

Example 1.—The elevations of C from A and B are α and β respectively, B is vertically above A and AB $= a$; find the height h of C above the level of A.

The exterior angle of the triangle BEC at E (Fig. 7) is α. Hence $\angle$BCA $= \alpha - \beta$. From the triangle ACB

$$\frac{\mathrm{AC}}{\sin (90^\circ + \beta)} = \frac{a}{\sin (\alpha - \beta)}.$$

Therefore $$\frac{h}{\sin\alpha\cos\beta} = \frac{a}{\sin(\alpha-\beta)},$$

which gives $\qquad h = a\sin\alpha\cos\beta\operatorname{cosec}(\alpha-\beta).$

Example 2.—The base line AB has length a, lies in the vertical plane through C and makes an angle θ with the horizontal. The elevations of C from A and B are α and β respectively. Find the height of C above the level of A.

From the triangle ABC (Fig. 8)

$$\frac{AC}{\sin ABC} = \frac{AB}{\sin BCA}.$$

Hence $$\frac{h/\sin\alpha}{\sin\{180^\circ-(\beta-\theta)\}} = \frac{a}{\sin(\beta-\alpha)}.$$

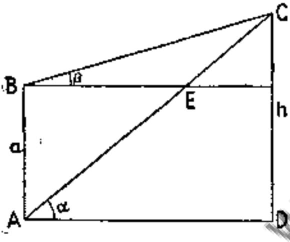

FIG. 7.

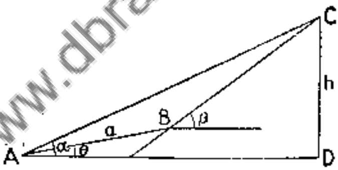

FIG. 8.

Therefore $\qquad h = a\sin\alpha\sin(\beta-\theta)\operatorname{cosec}(\beta-\alpha);$

a result which can be obtained from (6) by putting $\gamma = \alpha-\theta$ and $\delta = 180^\circ-(\beta-\theta)$.

Example 3.—From a point directly opposite the middle of one side of a square tower and distant d from it, the elevation of the top of the tower is observed to be α, and the elevation of the top of a flagstaff erected at the middle of the top of the tower is observed to be β. From another point on the same level, at a distance a from the first directly towards the tower, the top of the flagstaff can be seen and no more. Find the height of the flagstaff.

In Fig. 9 let $\angle DBC = \theta$.

Then $$\tan\theta = \frac{DC}{BD} = \frac{d\tan\alpha}{d-a}.$$

Also $$\frac{EF}{CD} = \frac{EC}{CB} = \frac{EC/CA}{CB/CA},$$

which gives, when we apply the Law of Sines to the triangles ACE and ABC,

$$\frac{h}{d \tan \alpha} = \frac{\sin (\beta - \alpha)/\sin (\theta - \beta)}{\sin \alpha/\sin \theta}.$$

Therefore $h = d \sin \theta \sin (\beta - \alpha) \sec \alpha \operatorname{cosec} (\theta - \beta)$, where $\tan \theta = d \tan \alpha/(d - a)$.

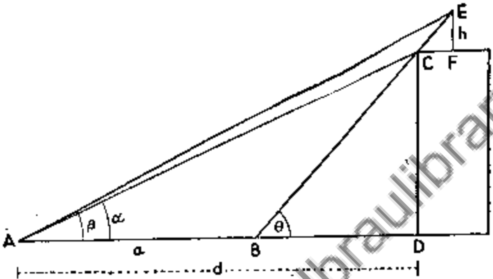

FIG. 9.

Example 4.—AB is a horizontal base line of length a. From A and B observations are made on a round tower, and the

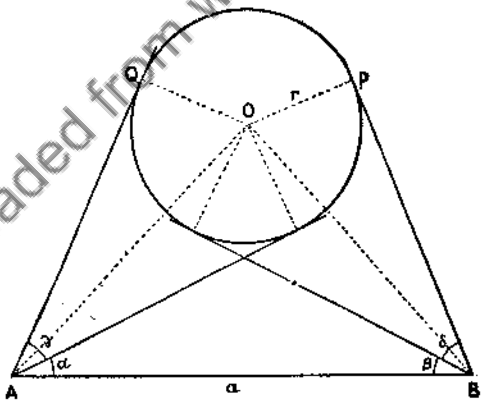

FIG. 10.

angles marked in Fig. 10, all of which lie in a horizontal plane, are measured. Find the radius r of the tower.

From the triangle OBP, $r = \text{OB} \sin \frac{1}{2}\delta$; and from the triangle OAB,

$$\frac{\text{OB}}{\sin(\alpha + \frac{1}{2}\gamma)} = \frac{\text{AB}}{\sin\{180^\circ - (\alpha + \frac{1}{2}\gamma + \beta + \frac{1}{2}\delta)\}}.$$

Therefore

$$r = a \sin \tfrac{1}{2}\delta \sin(\alpha + \tfrac{1}{2}\gamma) \operatorname{cosec}(\alpha + \beta + \tfrac{1}{2}\gamma + \tfrac{1}{2}\delta).$$

By a similar method, starting from $r = \text{OA} \sin \frac{1}{2}\gamma$, we obtain the equivalent result

$$r = a \sin \tfrac{1}{2}\gamma \sin(\beta + \tfrac{1}{2}\delta) \operatorname{cosec}(\alpha + \beta + \tfrac{1}{2}\gamma + \tfrac{1}{2}\delta).$$

Example 5.—A and B (Fig. 11) are points on the ground. By means of a theodolite T, placed vertically above A, the

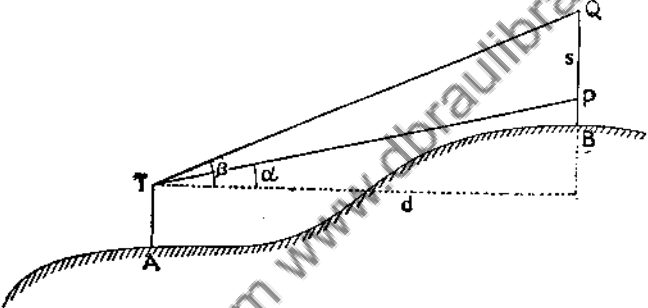

FIG. 11.

angles of elevation α, β of two marks P, Q on a staff held vertical at B are measured. If PQ $= s$, prove that the horizontal distance d of B from A is given by

$$d = \frac{s \cos \alpha \cos \beta}{\sin(\beta - \alpha)},$$

and that the rise in level h from A to B is given by

$$h = p + q - r,$$

where $$p = \frac{s \sin \alpha \cos \beta}{\sin(\beta - \alpha)},\ q = \text{AT},\ r = \text{BP}.$$

§ 6. Curvature of the Earth's Surface

Correction in calculating Heights.—In the problems of § 3 and in § 5, *Examples* 1, 2, no allowance has been made for the curvature of the earth, the distances involved being assumed not to be so great as to render correction for curvature necessary.

Let Fig. 12 represent a section through the centre O of the earth, the earth being considered as a perfect sphere of radius R.

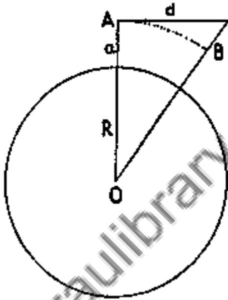

FIG. 12.

Let A be a point of observation whose height above the earth's surface is a; and let C be an object distant d from A, and observed to lie in the horizontal plane through A. Then AC is perpendicular to OA.

C is *apparently* on the same level as A; but the point B on OC, such that OB = OA is actually on the same level as A, that is, at the same height above the surface of the earth. Hence to the apparent height a of C must be added the correction BC in order that the true height may be obtained.

Now

$$\begin{aligned} BC = OC - OB &= \sqrt{\{(R+a)^2 + d^2\}} - (R+a) \\ &= (R+a)\left\{1 + \left(\frac{d}{R+a}\right)^2\right\}^{\frac{1}{2}} - (R+a) \\ &\doteqdot \frac{1}{2}\cdot\frac{d^2}{R+a}, \text{ since } \frac{d}{R+a} \text{ is very small,} \\ &\doteqdot \frac{d^2}{2R\left(1+\frac{a}{R}\right)} \\ &\doteqdot \frac{d^2}{2R}, \text{ since } \frac{a}{R} \text{ is very small.} \end{aligned}$$

Hence an object d miles away, which appears to be on the same level as the observer, is approximately $d^2/2R$ of a mile higher, R being the number of miles in the radius of the earth. Since R $\doteqdot$ 3960, this correction is $d^2/7920$ of a mile, or $\frac{2}{3}d^2$ feet. From this the surveyor obtains his working rule :

$$\text{Correction in feet} = \tfrac{2}{3}\,(\text{distance in miles})^2.$$

Example 1.—From a station A, 423 feet above mean sea-level, an object B, 21 miles distant, is observed through a horizontal telescope. What is the height of B above mean sea-level ?

$$\begin{aligned}\text{True height of B} &= \text{Height of A} + \text{correction for curvature}\\ &\doteqdot 423 + \tfrac{2}{3}\,.\,21^2 \text{ feet}\\ &\doteqdot 717 \text{ feet.}\end{aligned}$$

The Horizon.—The points of contact of tangents from A to the earth's surface (Fig. 13) lie on a circle whose plane is perpendicular to OA. This circle, which is the boundary of that part of the surface which is visible from A, is the *horizon* of the point A. The distance AT from A to any point on the circle is called the *distance of the horizon* from A. It is the maximum distance at which an object placed at A could be seen from the surface of the earth. The angle of depression $\angle$BAT from A of any point on the circle is called the *dip of the horizon* from A. If a is the height of A above the earth's surface, then

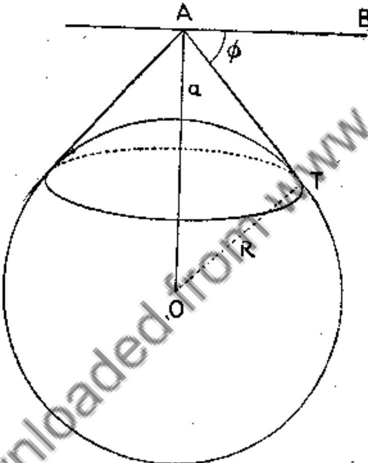

FIG. 13.

$$AT^2 = (R + a)^2 - R^2 = 2aR\left(1 + \frac{a}{2R}\right) \doteqdot 2aR,$$

since a/R is very small. Hence

$$\text{Distance of horizon from A} \doteqdot \sqrt{(2aR)}. \qquad (7)$$

If the height of A is a feet, the distance of the horizon $\fallingdotseq \sqrt{(2 \cdot \frac{a}{5280} \cdot 3960)}$ miles $= \sqrt{(\frac{3}{2}a)}$ miles; which gives the rule:

Distance of horizon in miles $\fallingdotseq \sqrt{(\frac{3}{2} \times \text{height in feet})}$. (8)

Again, if $\angle BAT = \phi$ radians, we have, since $\angle BAO$ and $\angle OTA$ are right,

$$\tan \phi = \tan T\hat{O}A = \frac{AT}{R} \fallingdotseq \sqrt{\left(\frac{2a}{R}\right)}, \text{ from (7).}$$

Hence, since ϕ is very small,* $\phi \fallingdotseq \sqrt{\left(\frac{2a}{R}\right)}$. Therefore

$$\text{Dip of horizon from A} \fallingdotseq \sqrt{\left(\frac{2a}{R}\right)} \text{ radians}$$

$$\fallingdotseq \frac{180}{\pi}\sqrt{\left(\frac{2a}{R}\right)} \text{ degrees.} \quad (9)$$

Note that, since $\phi \fallingdotseq \frac{AT}{R}$, the dip of the horizon in radians $\fallingdotseq$ (the distance of the horizon in miles) $\div$ 3960.

Example 2.—Find the distance d and the dip ϕ of the horizon from a point which is 720 feet above sea-level.

From (8), $d \fallingdotseq \sqrt{(\frac{3}{2} \times 720)} \fallingdotseq 32{\cdot}86$ miles.

Hence

$$\phi \fallingdotseq \frac{32{\cdot}86}{3960} \text{ of a radian}$$

$$\fallingdotseq \frac{32{\cdot}86 \times 180 \times 60}{3960\pi} \text{ minutes}$$

$$\fallingdotseq 28{\cdot}53 \text{ minutes.}$$

Therefore $d \fallingdotseq 32{\cdot}9$ miles and $\phi \fallingdotseq 28{\cdot}5$ minutes.

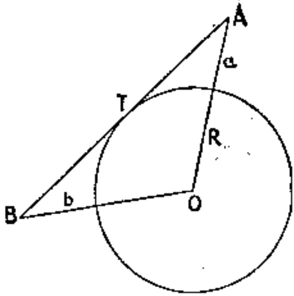

FIG. 14.

If an object B (Fig. 14) at height b above sea-level is just visible, beyond the horizon, from a point A at height a above sea-level, AB is a tangent, at the point T say, to the earth's surface.

* If ϕ is the number of radians in a very small angle, $\tan \phi \fallingdotseq \phi$.

Then $\quad AT \fallingdotseq \sqrt{(2aR)}\,;\ BT \fallingdotseq \sqrt{(2bR)}.$

Therefore the distance d from A to B is given by

$$d \fallingdotseq \sqrt{(2aR)} + \sqrt{(2bR)}\,;$$

or if a and b are expressed in feet,

$$d \fallingdotseq \sqrt{(\tfrac{3}{2}a)} + \sqrt{(\tfrac{3}{2}b)} \text{ miles} \quad . \quad . \quad (10)$$

Example 3.—The look-out of a ship, from a position 75 feet above the sea, can just see the flash of a lighthouse whose lamp is 182 feet above sea-level. How far is the ship from the lighthouse?

From (10), we obtain

$$\begin{aligned} d &\fallingdotseq \sqrt{(\tfrac{3}{2} \times 75)} + \sqrt{(\tfrac{3}{2} \times 182)} \\ &\fallingdotseq 10{\cdot}61 + 16{\cdot}52. \end{aligned}$$

The ship is therefore about 27·1 miles from the lighthouse.

EXAMPLES XI

1. A man standing on a level plain observes the elevation of a captive balloon to be 35°. He then walks 1000 yards towards the balloon, and finds that the elevation is now 52°. Determine the height of the balloon.

Ans. 1546 yards.

2. From a point on the bank of a river the elevation of a tree directly opposite on the other bank is 35° 12′; at a point 120 feet back from the first point and in line with that point and the tree the elevation is 16° 30′. Calculate the width of the river and the height of the tree.

Ans. 87 feet, 61 feet.

3. A person in a motor-boat which is sailing due south observes that a beacon lies in the direction 33° east of south. When the boat has covered a further distance of 358 yards, it is found that the direction of the beacon is 57° east of south. Assuming that the course of the boat remains unaltered, find at what distance from the beacon it will pass.

Ans. 402 yards.

4. A howitzer is firing shrapnel, and an observer at the gun notes that the burst takes place at an elevation of 22° 42′, while an observer at a distance of 310 yards right behind the gun finds that the elevation of the burst is 18° 18′. Find the height at which the shells are bursting.

Ans. 490 yards.

5. From a balloon the angles of depression of the top and the bottom of a tower whose height is h feet are observed to be α and β. Show that the height of the balloon above the ground and its horizontal distance from the tower are, respectively,

$$h \cos \alpha \sin \beta \operatorname{cosec} (\beta - \alpha), \quad h \cos \alpha \cos \beta \operatorname{cosec} (\beta - \alpha).$$

If $h = 112$ feet, $\alpha = 4° 30'$, $\beta = 5° 12'$, find the height of the balloon and its distance from the tower.

Ans. 828 feet, 9102 feet.

6. An observer finds that a captive balloon, situated due north of him, has an angle of elevation 33°, while a second observer, at a distance of 500 yards due west of the first observer, finds that the angle of elevation is 21°. Find the height of the balloon.

Ans. 238 yards.

7. A tower of height h, whose cross-section is a square with side of length 20 feet, stands on a level plain. An observer walks away from the tower in the line of one of the sides of the base until the elevations of the two corners which he can see are 54° and 43°. Find the height h.

Ans. 25·4 feet.

8. The angle of elevation of the top of a chimney from a place P due north is 2α, and from Q due east of P the angle of elevation is α. Show that the height of the chimney is given by

$$h = a\sqrt{\{\sin \alpha \sin^2 2\alpha \operatorname{cosec} 3\alpha\}},$$

where PQ is of length a, and P and Q lie in the same horizontal plane as the foot of the chimney.

Calculate h if $a = 350$ yards and $\alpha = 10° 30'$.

Ans. 74 yards.

9. A vertical flagstaff AT stands at a point A on one bank of a straight river; B and C are points on the other bank such that B is directly opposite A; and M is the mid-point of BC. If the angles of elevation of T, the top of the staff, from B, C and M are α, β and θ respectively, show that

$$4 \cot^2 \theta = 3 \cot^2 \alpha + \cot^2 \beta.$$

10. From a point at a height h above the level of the water in a loch the elevation of the top of a tree on the opposite side of the loch is α, and the angle of depression of the reflection of the top of the tree in the loch is β. Prove that the height of the top of the tree above the water is $h \sin (\beta + \alpha) \operatorname{cosec} (\beta - \alpha)$, and find its distance from the point of observation.

If h is 10 feet, α is $9^\circ\ 27'$, and β is $13^\circ\ 8'$, find the height of the tree above the water, and its distance from the point of observation.

Ans. $2h \cos\alpha \cos\beta \operatorname{cosec}(\beta - \alpha)$, 60 feet, 299 feet.

11. From a window of a house on one side of a street the angles of elevation and depression of the top and the foot of the house opposite are found to be α and β respectively. If k is the height above the street of the point from which the angles are measured, and if h is the height of the house opposite, show that

$$h = k \sin(\alpha + \beta) \sec\alpha \operatorname{cosec}\beta.$$

If $k = 25$ feet, $\alpha = 33^\circ\ 18'$, $\beta = 37^\circ\ 42'$, determine h and find the breadth of the street.

Ans. 46 feet, 32 feet.

12. Looking out of a window, with an eye at the height of 15 feet above the roadway, an observer finds that the angle of elevation of the top of a telegraph post is $17^\circ\ 18'$, and that the angle of depression of the foot of the post is $8^\circ\ 30'$. Calculate the height of the telegraph post, and its distance from the observer.

Ans. 46 feet, 100 feet.

13. A and B are two points a distance d apart, and C is the top of a hill; the angles BAC and CBA are α and β respectively, and AC makes an angle γ with the horizontal. Prove that the height of C above A is

$$d \sin\beta \sin\gamma \operatorname{cosec}(\alpha + \beta).$$

Calculate this height when $d = 880$ yards, $\alpha = 29^\circ$, $\beta = 58^\circ$, $\gamma = 4^\circ\ 16'$.

Ans. 167 feet.

14. The height of the top of a flagstaff, situated at the edge of a cliff, above the top of the cliff is h feet, and the angle subtended by the flagstaff at a boat is α; if the angle of elevation of the top of the cliff from the boat is β, show that the height of the cliff and the distance of the boat from the bottom of the cliff in feet are, respectively,

$$h \sin\beta \cos(\alpha + \beta) \operatorname{cosec}\alpha, \quad h \cos\beta \cos(\alpha + \beta) \operatorname{cosec}\alpha.$$

If $h = 34$, $\alpha = 3^\circ\ 8'$, $\beta = 26^\circ\ 52'$, find the height of the cliff, and the distance of the boat from the foot of the cliff.

Ans. 243 feet, 481 feet.

15. A lighthouse on the edge of a cliff is viewed from a boat. The angle of elevation of the top of the lighthouse is α. After the boat has been rowed x yards directly towards the lighthouse it is found that the elevations of the top and foot of the

lighthouse are β and γ respectively. If h feet and k feet are the heights of the cliff and the lighthouse respectively, prove that

(i) $h = 3x \sin\alpha \cos\beta \tan\gamma \operatorname{cosec}(\beta - \alpha)$,

(ii) $k = 3x \sin\alpha \sin(\beta - \gamma) \sec\gamma \operatorname{cosec}(\beta - \alpha)$.

If $x = 27$, $\alpha = 33° 18'$, $\beta = 62° 36'$, $\gamma = 46° 24'$, find h and k.

Ans. 44, 37.

16. A cliff with a tower on its edge is observed from a boat at sea, and the elevation of the top of the tower is 18°; after the boat has been rowed a distance of 500 yards towards the tower, the elevations of the top and the bottom of the tower are found to be 42° and 39° respectively. Calculate the height of the tower.

Ans. 77 feet.

17. A person at sea-level observes that the angle of elevation of the summit of a mountain is 32° 42′. After walking a distance of 350 yards up a road which runs directly towards the mountain, and which rises uniformly, making an angle of 5° with the horizontal, he finds that the angle of elevation of the summit is 36° 48′. Find the height of the mountain.

Ans. 4181 feet.

18. An observer on the shore of a lake finds that the angle of elevation of the summit of a mountain on the opposite shore of the lake is α. He then ascends a mountain railway which rises uniformly, making an angle γ with the horizontal, in a direction opposite to the direction of the summit of the mountain, till he reaches a station at a height k above the level of the lake. The angle of elevation of the summit of the mountain is then found to be β. If h is the height of the mountain above the lake, show that

$$h = k \sin\alpha \sin(\beta + \gamma) \operatorname{cosec}\gamma \operatorname{cosec}(\alpha - \beta).$$

If $\alpha = 47° 18'$, $\beta = 33° 36'$, $\gamma = 22°$, $k = 653$ feet, find h.

Ans. 4463 feet.

19. From a point A on a level higher than that of the foot of a monument it is found that the angle of elevation of the top of the monument is α. A path runs downhill from A in the direction of the monument to a point B on the same level as the foot of the monument, and at B the angle of elevation of the top of the monument is found to be β. If h is the height of the monument, x the distance AB, and γ the angle made by BA with the horizontal, prove that

$$h = x \sin\beta \sin(\alpha + \gamma) \operatorname{cosec}(\beta - \alpha).$$

If $x = 42$ yards, $\alpha = 22° 42'$, $\beta = 54° 18'$, and $\gamma = 13° 30'$, find the height of the monument in feet.

Ans. 115.

20. From a position P a person finds that the elevation of the top of a spire is α. He then walks downhill from P in a direction directly away from the spire till he reaches a position Q. From this point the elevations of the top of the spire and of P are found to be β and γ respectively. If h is the height of the spire above P, and d the height of P above Q, prove that

$$d = h \sin \gamma \sin (\alpha - \beta) \operatorname{cosec} \alpha \operatorname{cosec} (\beta - \gamma).$$

If $h = 150$ feet, $\alpha = 73° \ 36'$, $\beta = 54° \ 42'$, $\gamma = 33° \ 12'$, find d.

Ans. 76 feet.

21. The southern part of the wall of a city runs due east and west, and is of uniform height. An observer situated on the plain which lies to the south of the wall finds that the elevation of the part of the wall nearest to him is α. After walking in a direction θ north of east for a distance d, he finds that the elevation of the part of the wall now nearest is β. If h is the height of the wall, show that

$$h = d \sin \theta \sin \alpha \sin \beta \operatorname{cosec} (\beta - \alpha).$$

If $\alpha = 33° \ 18'$, $\beta = 74° \ 42'$, $\theta = 45°$, $d = 56$ feet, find the value of h correct to the nearest foot.

Ans. 32 feet.

22. A lightship is at a distance 5 miles due south of a point A; a launch starts from the lightship in a direction 10° east of south, and at the end of half an hour alters its course to due east. At the end of another half-hour the bearing of the launch from A is 31° east of south; find the speed of the launch in miles per hour.

Ans. 10·3.

23. At a point A, on the same level as P, the foot of a vertical tower PQ, the angle of elevation of the top of the tower is α. At a point B, distant a from A, and such that the angle PAB is β, AP subtends an angle γ. Prove that the height of the tower is

$$a \tan \alpha \sin \gamma \operatorname{cosec} (\beta + \gamma).$$

From the data $\alpha = 9° \ 18'$, $\beta = 60°$, $a = 750$ feet and $\gamma = 79° \ 23'$, calculate the height of the tower.

Ans. 185 feet.

24. From a point 2000 feet above sea level a speedboat was observed due east, its angle of depression being 8° 12′. From the same point, 1 minute 30 seconds later, the boat was observed 15° south of east, its angle of depression being 11° 41′. Assuming that a straight course was steered, find the direction

of the course, and, in miles per hour, the average speed of the boat between the times of observation.

Ans. 28° 53′ south of west; 39·3 miles per hour.

25. From a fixed station S three observations are taken, at equal intervals, of the bearing of a ship steaming at uniform speed in a straight course. If these bearings are, in order, α north of east, due east, and β south of east, and the direction of the ship's course is θ south of east, prove that

$$\tan\theta = 2\sin\alpha\sin\beta\operatorname{cosec}(\alpha-\beta).$$

If $\alpha = 37° \ 20'$ and $\beta = 25° \ 45'$, calculate θ.

Ans. 69° 8′.

26. From a ship which is sailing due south to north a mountain-top is seen in a direction θ degrees north of west, and the angle of elevation is found to be α. When the summit is due west its angle of elevation is found to be β. If h is the height of the mountain and d the distance that the ship has sailed between the two observations, show that

$$\text{(i)}\ \cos\theta = \tan\alpha\cot\beta, \quad \text{(ii)}\ h = d\tan\beta\cot\theta.$$

If $d = 2750$ yards, $\alpha = 14° \ 12'$, $\beta = 19° \ 36'$, find θ and h.

Ans. 44° 43′, 2967 feet.

27. Two straight roads ABC and AX intersect at an angle CAX $= \beta$; two houses are situated at B and C, and AB $= x$ feet, BC $= y$ feet. A man walking along AX observes the angle subtended by BC, and finds that it is greatest at a certain point P in AX: prove that the circle through B, C and P touches AX at P, and deduce that, if $\angle$BPC $= \alpha$,

$$\text{(i)}\ \frac{x}{y} = \frac{\cos^2\frac{1}{2}(\alpha+\beta)}{\sin\alpha\sin\beta}; \quad \text{(ii)}\ \text{AP} = x\frac{\cos\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}(\alpha+\beta)}.$$

28. A flagstaff of height h stands on top of a tower, and subtends an angle α at all points on the horizontal plane through the foot of the tower which are distant either a or b from the tower. Show that

$$h = (a+b)\tan\alpha.$$

29. Four points A, B, C, D lie in a horizontal plane; A and B are 300 yards apart, and by measurement it is found that $\angle$CBA $= 43° \ 40'$; $\angle$BAC $= 110° \ 15'$; $\angle$DBA $= 92° \ 10'$; $\angle$BAD $= 49° \ 18'$. Calculate the length of CD.

Ans. 1449 feet.

30. From A, B, points of observation 100 yards apart in a horizontal plane, the following angles are obtained by measurement, the points C, D being in the same horizontal plane with A, B: $\angle$BAC $= 118° \ 15'$; $\angle$CBA $= 41° \ 36'$; $\angle$BAD $= 50° \ 46'$; $\angle$DBA $= 92° \ 18'$. Calculate the length of CD.

Ans. 602 feet.

31. A and B are two stations 1200 feet apart; P and Q are two points in a plane through AB and on the same side of AB. With the following data calculate PA, QA, PB, QB, PQ: $\angle PAB = 73° \ 27'$; $\angle ABP = 31° \ 14'$; $\angle QAB = 42° \ 11'$; $\angle ABQ = 87° \ 10'$.

Ans. 643 feet, 1550 feet, 1189 feet, 1042 feet, 1054 feet.

32. A and B are two signalling stations, A being 20 miles due north of B. At a certain instant a ship is observed bearing 58° 48′ east of south from A and 77° 28′ east of north from B. An hour later the ship bears due east from A and 31° 17′ east of north from B. Calculate the distance of the ship from A at each of the instants of observation, the speed of the ship and the direction of its course.

Ans. 28·2 miles, 12·2 miles, 18·9 miles per hour, 39° 22′ west of north.

33. At a point A the bearing of P is 27° 20′ and the bearing of R is 56° 45′. On moving a distance of 25 miles to B in a direction whose bearing is 84° 12′, the bearing of P is 315° 28′ and the bearing of R is 16° 5′. Find in miles the distances of R from A and from P. [Each bearing is measured from the north clockwise.]

Ans. 35·6, 20·4.

34. Two inaccessible points P and Q are observed from two stations, the second of which is 500 yards due east of the first. From the first station P bears due north and Q 40° east of north. From the second station P bears 55° west of north and Q 20° east of north. Find the distances of Q east and north of P, and the bearing of Q as seen from P.

Ans. 883 yards, 702 yards, 38° 30′ north of east.

35. A point A lies 800 yards due north of a point B. The point P bears 55° 17′ west of south from A, and 72° 20′ west of south from B. The point Q is due east of P, and bears 38° 33′ east of south from A. Find the distances AP and PQ, and the bearing of Q from B.

Ans. 2600 yards, 3317 yards, 29° 59′ south of east.

36. A and B are two points on one bank of a river, and P and Q are two points on the opposite bank, such that PQ and AB are parallel and in the same sense. If $AB = a$, $\angle PAQ = \alpha$, $\angle QAB = \beta$, and $\angle ABP = \gamma$, prove that

$$PQ = a \sin \alpha \sin \gamma \operatorname{cosec} \beta \operatorname{cosec} (\alpha + \beta + \gamma).$$

Calculate PQ when $\alpha = 35° \ 22'$, $\beta = 24° \ 7'$, $\gamma = 45° \ 40'$ and $a = 200$ feet. Find also the width of the river.

Ans. 210 feet, 128 feet.

37. A, B, C are any points on a directed line and P is a point not on the line. If AP, BP, CP make angles α, β, γ respectively with the positive direction of the line, show that

$$BC \cot \alpha + CA \cot \beta + AB \cot \gamma = 0.$$

From a ship steaming due east at uniform speed a lighthouse is observed on a bearing of 27° 13′ south of east. An hour later the bearing of the lighthouse is 52° 29′ south of east. What will the bearing of the lighthouse be after two hours further steaming?

Ans. W. 32° 14′ S.

38. A ship steaming at m miles per hour in a direction S. 35° E. is due west of a lighthouse at noon. At 12.20 p.m. it is due south of the lighthouse. What is its bearing from the lighthouse at 12.50 p.m.? How far is the ship from the lighthouse at this time if $m = 12$?

Ans. S. 22° 47′ E., 8·9 miles.

39. A person on a ship sailing north sees two lighthouses, which are 14 miles apart, in a line due west; after two hours sailing one of them bears S. 22° W. and the other S. 47° 15′ W. Find the ship's rate.

Ans. 10·3 miles per hour.

40. From the deck of a steamer two lighthouses, whose distance apart is known to be 5·8 miles, are seen in the directions N. 37° E. and N. 62° E. The steamer sails due north until the lighthouses are observed to be in line, in the direction S. 49° E. If the steamer has been sailing for 1 hour 20 minutes between the observations, find its speed in miles per hour.

Ans. 12·7.

41. From a ship steaming in a south-easterly direction at the rate of 21½ miles an hour, a lighthouse is observed at noon bearing S. 20° 40′ E. Half an hour later the lighthouse is observed to bear S. 25° 16′ W. At what time will the ship be at its shortest distance from the lighthouse, and what will that distance be?

Ans. 12 hours 36 minutes p.m., 5·8 miles.

42. From a ship whose course is W. 30° S. two lighthouses 6 miles apart are seen in a line due west. Twenty-five minutes later one of them bears N. 31° E. and the other N. 63° W. Find the speed of the ship in miles per hour.

Ans. 11·2.

43. Two steamers A and B sail from two points P and Q, respectively, which are 12 miles apart, P being due north of Q. If the A's course is due east, B's is 40° to the north of east.

steamers start simultaneously at midday, and have the same speed, 8 miles per hour, find:

(i) their distance apart at 1.30 p.m.;
(ii) the time at which they will be nearest to each other;
(iii) the distance which will then separate them.

Ans. (i) 5·1 miles, (ii) 2 hours 4 minutes p.m., (iii) 4·1 miles.

44. A coastguard station B is 10 miles due east of another station A. At noon a ship is observed S. 30° W. from A and S. 65° W. from B. The ship is heading due east, but a current is flowing in a direction S. 60° W., so that, after two hours, the ship bears S. 60° E. from A and S. 20° E. from B. Find (i) the distance between the two positions of the ship, (ii) the actual speed of the ship through the water.

Ans. (i) 16·4 miles, (ii) 9·0 miles per hour.

45. A cruiser A, whose speed is 25 miles per hour, learns that an enemy ship B distant 25 miles in a direction due east is steaming N. 30° E. at 20 miles per hour. The range of A's guns is 10 miles. Find the course A should steer to get B within range as quickly as possible, and the time it will take to achieve this.

Ans. E. 43° 51′ N., 1 hour 52 minutes.

46. To determine the height of a hill a base line AB = 1250 feet is measured in a direction E. 33° 12′ N. From B the summit bears N. 48° 12′ W., and has an elevation of 18° 4′. From A the summit bears N. 20° 15′ E. Find the height of the hill.

Ans. 261 feet.

47. B is the foot of a vertical pole AB. From C, a point due south of B in the same horizontal plane, the angle of elevation of the top A is α. A man starts from C and walks in the direction θ east of north to a point D, where CD is equal to m yards. The angle of elevation of A from D is observed to be β. Show that if ϕ is the angle BDC, then $\sin\phi = \sin\theta\cot\alpha\tan\beta$, and the height of the pole is $m\sin\theta\tan\beta\operatorname{cosec}(\theta+\phi)$ yards.

Given that $\alpha = 8°\ 10'$, $\beta = 7°\ 30'$, $\theta = 72°\ 15'$, $m = 360$, find the height of the pole.

Ans. 61·9 yards.

48. CD is a vertical mast. A and B are points on the horizontal plane through C, the foot of CD, A bearing due west from C and B bearing due south from C. The elevation of D from A is 28°, and from B is 32°, and AB is 123 feet in length. Find the height of CD.

Ans. 49·8 feet.

49. A point P, whose projection on a horizontal plane is Q, is observed from points A, B on the horizontal plane, 1255 yards apart. If the angle AQB is 22° 30′, and if the angles of elevation of P at A and B are 12° 16′ and 10° 53′ respectively, calculate the height of QP.

Ans. 627 yards.

50. From three points A, B and C on a straight level road the angles of elevation of a hilltop are θ, θ and ϕ respectively. If AB = BC = 800 feet, prove that the height of the hill is

$$\frac{800 \cdot \sqrt{2} \cdot \sin\theta \sin\phi}{\sqrt{\{\sin(\theta+\phi)\sin(\theta-\phi)\}}} \text{ feet.}$$

Calculate the height when $\theta = 16° \, 36'$ and $\phi = 13° \, 24'$.

Ans. 448 feet.

51. Two mutually perpendicular straight lines OAB, OPQ, drawn from a point O outside a given circle, cut the circle in A, B and P, Q respectively. Prove that

$$PQ = a \cos(\alpha+\beta) \operatorname{cosec}(\alpha-\beta),$$

where $\angle PAO = \alpha$, $\angle PBO = \beta$ and $AB = a$.

The angles of elevation of the top of a tower from two points A and B, due south of the tower and 189 feet apart, are 48° 15′ and 39° 48′. A vertical flagstaff on the tower subtends equal angles at A and B. Calculate the length of the flagstaff.

Ans. 43·8 feet.

52. An aeroplane is seen flying north-west at a uniform distance from the ground. An observer finds that when it is seen in a direction θ east of north its elevation is α, and when it is seen ϕ west of north, its elevation is β. Show that

$$\tan\beta(\cos\theta + \sin\theta) = \tan\alpha(\cos\phi - \sin\phi).$$

CHAPTER XII

PROPERTIES OF QUADRILATERALS

§ 1. Notation

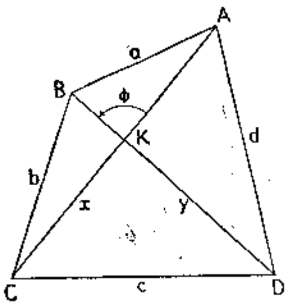

FIG. 1.

LET the quadrilateral ABCD (Fig. 1) be convex. The magnitudes of the angles at A, B, C and D are denoted by A, B, C and D, and the lengths of the sides AB, BC, CD, DA by a, b, c, d respectively. The lengths of the diagonals AC and BD are denoted by x and y, the area enclosed by the quadrilateral by S, and the semi-perimeter by s, so that $2s = a + b + c + d$.

§ 2. Area of a Quadrilateral

If $A + C = 2\alpha$, then

$$S^2 = (s - a)(s - b)(s - c)(s - d) - abcd \cos^2 \alpha \quad . \quad (1)$$

For (Fig. 1)

$$y^2 = a^2 + d^2 - 2ad \cos A = b^2 + c^2 - 2bc \cos C,$$

so that

$$2(ad \cos A - bc \cos C) = a^2 + d^2 - b^2 - c^2.$$

Also, since the area of the quadrilateral is the sum of the areas of the triangles ABD and CDB

$$2(ad \sin A + bc \sin C) = 4S.$$

Now square and add corresponding sides of these two equations, and get

$$4(a^2d^2 + b^2c^2 - 2abcd \cos 2\alpha)$$
$$= 16S^2 + (a^2 + d^2 - b^2 - c^2)^2.$$

or

$$4(ad + bc)^2 - 16abcd \cos^2 \alpha = 16S^2 + (a^2 + d^2 - b^2 - c^2)^2.$$

Hence

$$\begin{aligned}16S^2 &= \{2(ad + bc) + (a^2 + d^2 - b^2 - c^2)\} \\ &\quad \times \{2(ad + bc) - (a^2 + d^2 - b^2 - c^2)\} - 16abcd \cos^2 \alpha \\ &= \{(a + d)^2 - (b - c)^2\}\{(b + c)^2 - (a - d)^2\} \\ &\qquad - 16abcd \cos^2 \alpha \\ &= (a + d + b - c)(a + d - b + c) \\ &\quad \times (b + c + a - d)(b + c - a + d) - 16abcd \cos^2 \alpha \\ &= 16(s - c)(s - b)(s - d)(s - a) - 16abcd \cos^2 \alpha,\end{aligned}$$

which gives formula (1).

COROLLARY.—If ABCD is a cyclic quadrilateral, $A + C = 180°$ and $\cos \alpha = 0$, so that

$$S = \sqrt{\{(s - a)(s - b)(s - c)(s - d)\}} \quad . \quad (2)$$

Expression for the Area in Terms of the Diagonals and an Included Angle.—Let the diagonals AC and BD (Fig. 1) intersect in K, and let ϕ denote the angle AKB: then

$$S = \tfrac{1}{2}xy \sin \phi. \quad . \quad . \quad . \quad (3)$$

For

$$\begin{aligned}\triangle KAB + \triangle KBC &= \tfrac{1}{2}AK \,.\, BK \sin \phi + \tfrac{1}{2}BK \,.\, KC \sin \phi \\ &= \tfrac{1}{2}x \,.\, BK \sin \phi,\end{aligned}$$

and

$$\begin{aligned}\triangle KCD + \triangle KDA &= \tfrac{1}{2}KC \,.\, KD \sin \phi + \tfrac{1}{2}KD \,.\, AK \sin \phi \\ &= \tfrac{1}{2}x \,.\, KD \sin \phi.\end{aligned}$$

Hence, on addition

$$S = \tfrac{1}{2}x\,(BK + KD) \sin \phi = \tfrac{1}{2}xy \sin \phi.$$

Expression for the Area in Terms of the Sides and Diagonals.—The law of cosines, when applied to the triangles KAB, KBC, KCD, KDA, gives

$$\begin{aligned}2\,AK \,.\, BK \cos \phi &= AK^2 + BK^2 - a^2, \\ 2\,BK \,.\, KC \cos \phi &= b^2 - BK^2 - KC^2, \\ 2\,KC \,.\, KD \cos \phi &= KC^2 + KD^2 - c^2, \\ 2\,KD \,.\, AK \cos \phi &= d^2 - KD^2 - AK^2.\end{aligned}$$

Hence, on addition,

$$2xy \cos \phi = -a^2 + b^2 - c^2 + d^2. \quad . \quad (4)$$

But, from (3),

$$2xy \sin \phi = 4S.$$

If corresponding sides of these two equations are squared and added, they give

$$4x^2y^2 = 16S^2 + (-a^2 + b^2 - c^2 + d^2)^2,$$

whence $$S = \tfrac{1}{4}\sqrt{\{4x^2y^2 - (a^2 - b^2 + c^2 - d^2)^2\}} \quad . \quad (5)$$

Example 1.—By eliminating S between (1) and (5), prove that

$$x^2y^2 = a^2c^2 + b^2d^2 - 2abcd \cos 2\alpha.$$

Example 2.—From (3) and (4) deduce that

$$S = \tfrac{1}{4}(-a^2 + b^2 - c^2 + d^2) \tan \phi.$$

§ 3. Diagonals of a Cyclic Quadrilateral

From the triangles ABC and CDA (Fig. 1).

$$x^2 = a^2 + b^2 - 2ab \cos B,$$

and $$x^2 = c^2 + d^2 - 2cd \cos D.$$

But, if the quadrilateral is cyclic, $\cos D = -\cos B$; hence

$$x^2(cd + ab) = (a^2 + b^2)cd + (c^2 + d^2)ab$$
$$= (ac + bd)(ad + bc).$$

Therefore $$x^2 = \frac{(ac + bd)(ad + bc)}{(ab + cd)}. \quad . \quad . \quad (6)$$

Similarly $$y^2 = \frac{(ab + cd)(ac + bd)}{(ad + bc)}.$$

It follows, on multiplying corresponding sides of these equations and taking the square roots of the products, that

$$xy = ac + bd. \quad . \quad . \quad . \quad (7)$$

This formula gives the theorem in geometry known as *Ptolemy's Theorem.*

Example 1.—Show that $\dfrac{x}{y} = \dfrac{ad + bc}{ab + cd}$.

Example 2.—Deduce Ptolemy's Theorem from § 2, *Example* 1.

§ 4. Radius of the Circum-circle of a Cyclic Quadrilateral

On applying the formula $4R\triangle = abc$ to the triangles ABC and ACD (Fig. 1) it is found that

$$4R \times \triangle ABC = abx,$$

and

$$4R \times \triangle ACD = cdx.$$

Hence, by addition,

$$4RS = (ab + cd)x = \sqrt{\{(ab + cd)(ac + bd)(ad + bc)\}};$$

so that

$$R = \tfrac{1}{4}\sqrt{\left\{\frac{(ab + cd)(ac + bd)(ad + bc)}{(s - a)(s - b)(s - c)(s - d)}\right\}} \quad . \quad (8)$$

§ 5. Formulæ for the Angles of a Cyclic Quadrilateral

From triangles ABD and BCD we find that

$$2ad \cos A = a^2 + d^2 - y^2,$$

and

$$2bc \cos C = b^2 + c^2 - y^2.$$

Hence, on subtraction, since $\cos C = -\cos A$,

$$2(ad + bc) \cos A = a^2 - b^2 - c^2 + d^2,$$

or

$$\cos A = \frac{a^2 - b^2 - c^2 + d^2}{2(ad + bc)} \quad . \quad . \quad (9)$$

Again, the areas of the triangles ABD and BCD are $\frac{1}{2}ad \sin A$ and $\frac{1}{2}bc \sin C$. Hence, on adding, since $\sin C = \sin A$,

$$S = \tfrac{1}{2}(ad + bc) \sin A,$$

so that

$$\sin A = \frac{2S}{ad + bc} \quad . \quad . \quad . \quad (10)$$

Again, from the formula

$$2(ad + bc) \cos A = a^2 - b^2 - c^2 + d^2,$$

it follows that

$$\begin{aligned} 4(ad + bc) \cos^2 \tfrac{1}{2}A &= 2(ad + bc)(1 + \cos A) \\ &= 2(ad + bc) + a^2 - b^2 - c^2 + d^2 \\ &= (a + d)^2 - (b - c)^2 \\ &= (a + d + b - c)(a + d - b + c). \end{aligned}$$

Thus $$\cos \tfrac{1}{2}\text{A} = \sqrt{\left\{\frac{(s-b)(s-c)}{ad+bc}\right\}} \quad . \quad . \quad (11)$$

Similarly

$$\begin{aligned} 4(ad+bc)\sin^2 \tfrac{1}{2}\text{A} &= 2(ad+bc)(1-\cos \text{A}) \\ &= 2(ad+bc) - (a^2-b^2-c^2+d^2) \\ &= (b+c)^2 - (a-d)^2 \\ &= (b+c+a-d)(b+c-a+d). \end{aligned}$$

Therefore $$\sin \tfrac{1}{2}\text{A} = \sqrt{\left\{\frac{(s-d)(s-a)}{ad+bc}\right\}} \quad . \quad . \quad (12)$$

Hence, from (11) and (12),

$$\tan \tfrac{1}{2}\text{A} = \sqrt{\left\{\frac{(s-d)(s-a)}{(s-b)(s-c)}\right\}} \quad . \quad . \quad (13)$$

Example.—Show that

$$(s-b)\tan \tfrac{1}{2}\text{A} = (s-d)\tan \tfrac{1}{2}\text{B}.$$

§ 6. Quadrilateral Circumscribed to a Circle

If a circle can be inscribed in the quadrilateral ABCD (Fig. 2),

$$a + c = b + d \quad . \quad . \quad . \quad (14)$$

For, if P, Q, R, S are the points of contact of AB, BC, CD, DA with the inscribed circle,

$$a = \text{AP} + \text{PB} = \text{SA} + \text{BQ},$$

and $$c = \text{CR} + \text{RD} = \text{QC} + \text{DS}.$$

Therefore $a + c = (\text{BQ} + \text{QC}) + (\text{DS} + \text{SA}) = b + d.$

Conversely, if $a + c = b + d$, a circle can be inscribed in the quadrilateral.

If, as in Fig. 3, $a < b$, cut off BE from BC equal to AB.

Then, since $$a + c = b + d$$
$$c = \text{EC} + d,$$

so that $d < c$. From DC cut off DF equal to DA. Then

$$\text{CF} = \text{EC}.$$

Now join AE, EF and FA. Since the triangles BEA, CFE and DAF are isosceles, the bisectors of the angles B, C and D bisect AE, EF and FA at right angles. Consequently they intersect in the circum-centre I of the triangle AEF. But I is equidistant from AB, BC, CD and DA. Hence a circle can be drawn with I as centre to touch the sides of the quadrilateral ABCD.

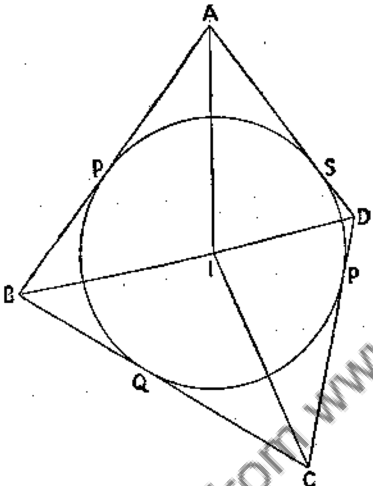

FIG. 2.

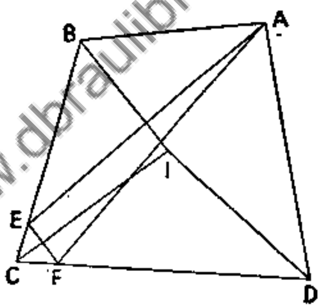

FIG. 3.

Area of the Quadrilateral.—From (1)

$$S^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\alpha,$$

where

$$\alpha = \tfrac{1}{2}(A + C).$$

But, since $a + c = b + d$,

$$s = a + c = b + d.$$

Hence

$$s - a = c, \quad s - b = d, \quad s - c = a, \quad s - d = b, \qquad (15)$$

so that

$$S^2 = abcd - abcd\cos^2\alpha$$
$$= abcd\sin^2\alpha.$$

Therefore

$$S = \sqrt{(abcd)}\sin\alpha \quad . \quad . \quad . \quad (16)$$

In particular, if the quadrilateral is cyclic, $\alpha = 90°$ and

$$S = \sqrt{(abcd)} \quad . \quad . \quad . \quad (17)$$

Radius of the Inscribed Circle.—If I (Fig. 2) is the centre and r the radius of the inscribed circle,

$$\begin{aligned} S &= \triangle IAB + \triangle IBC + \triangle ICD + \triangle IDA \\ &= \tfrac{1}{2}ra + \tfrac{1}{2}rb + \tfrac{1}{2}rc + \tfrac{1}{2}rd = rs, \end{aligned}$$

and therefore $$r = \frac{S}{s} \quad . \quad . \quad . \quad . \quad (18)$$

Example 1.—If a circle can be inscribed in the cyclic quadrilateral ABCD, show that

$$\tan \tfrac{1}{2}A = \sqrt{\left(\frac{bc}{ad}\right)}.$$

Example 2.—If the quadrilateral ABCD is inscribed in a circle of radius R, and circumscribed to a circle of radius r, show that

(i) $r^2(a + c)^2 = abcd,$
(ii) $16R^2\, abcd = (ab + cd)(ac + bd)(ad + bc).$

EXAMPLES XII

1. The sides of a quadrilateral, taken in order, are of lengths 3, 2, 2, 1, and the angle between the first two is 60°. Show that the quadrilateral is cyclic, and that $R = \frac{1}{3}\sqrt{(21)}$.

2. In formulæ (9) to (13) for the cyclic quadrilateral let $c \to 0$ and deduce corresponding formulæ for the triangle ABC.

3. The sides AB and CD of the cyclic quadrilateral ABCD are produced to meet in P. Show that

$$PA \,.\, PB = \frac{a^2bdxy}{(by - dx)^2}.$$

If BC and DA meet in Q, deduce that

$$PQ^2 = (ad + bc)(ab + cd)\left\{\frac{bd}{(b^2 - d^2)^2} + \frac{ac}{(a^2 - c^2)^2}\right\}.$$

[*Note.*—It can be proved geometrically that
$$PQ^2 = PA \,.\, PB + QB \,.\, QC.]$$

4. If the diagonals of a cyclic quadrilateral ABCD intersect in K, show that

$$AK = \frac{adx}{ad + bc},$$

and deduce that

$$AK \,.\, KC = \frac{abcd(ac + bd)}{(ab + cd)(ad + bc)}.$$

5. With the notation of the previous example, prove that the distance from the centre of the circle to K is

$$\frac{R}{(ab+cd)(ad+bc)}[(ac+bd)\{ac(b^2-d^2)^2+bd(a^2-c^2)^2\}]^{\frac{1}{2}}.$$

[*Note.*—If O is the centre, $OK^2 = R^2 - AK \cdot KC$.]

6. Show that, for the cyclic quadrilateral ABCD,

$$\cos\phi = \frac{b^2+d^2-a^2-c^2}{2(ac+bd)}, \quad \tan\tfrac{1}{2}\phi = \sqrt{\left\{\frac{(s-b)(s-d)}{(s-a)(s-c)}\right\}},$$

where ϕ is one of the angles between the diagonals.

7. Show that, in a cyclic quadrilateral ABCD,

$$\tan\psi = \frac{4S}{(b^2-d^2)(a^2-c^2)}\cdot\frac{(ad+bc)(ab+cd)}{ac+bd},$$

where ψ is one of the angles between the straight lines joining the mid-points of the opposite sides.

[Apply (3) and (4) to the parallelogram whose vertices are the mid-points of the sides of the quadrilateral, and then make use of (6).]

8. If K, L, M are the points of intersection of the diagonals of a quadrilateral ABCD inscribed in a circle, show that the area of the triangle KLM is to the area of the quadrilateral in the ratio $4a^2b^2c^2d^2 : |(a^2b^2-c^2d^2)(b^2c^2-d^2a^2)|$.

9. A quadrilateral ABCD is circumscribed to a circle: show that

$$AB\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B = CD\sin\tfrac{1}{2}C\sin\tfrac{1}{2}D.$$

10. If a circle can be inscribed in the quadrilateral ABCD, show that

$$S = \tfrac{1}{2}\{x^2y^2-(ac-bd)^2\}^{\frac{1}{2}}.$$

11. Show that, if the quadrilateral ABCD can be circumscribed to a circle,

$$ad\sin^2\tfrac{1}{2}A = bc\sin^2\tfrac{1}{2}C.$$

12. A cyclic quadrilateral ABCD is such that a circle can be inscribed in it; prove that

$$\cos A = (ad-bc)/(ad+bc).$$

13. A polygon of $2n$ sides, n of which are equal to a, and n equal to b, is inscribed in a circle; show that the radius of the circle is

$$\tfrac{1}{2}\sqrt{(a^2+2ab\cos\frac{\pi}{n}+b^2)}\cdot\operatorname{cosec}\frac{\pi}{n}.$$

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